Some combinatorial cases of the three matrix analog of Gerstenhaber's theorem

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Abstract Let k be a field. By a theorem of Gerstenhaber from the early 1960s, the unital k-algebra generated by two pairwise commuting $d \times d$ matrices with entries in k is a finite dimensional k-vector space of dimension at most d. The analog of this theorem for four or more pairwise commuting matrices is false. The three matrix version remains open. In this paper, we use combinatorial and commutative-algebraic methods to prove that the three matrix analog of Gerstenhaber's theorem holds for some infinite families of examples, each of which is combinatorial in nature.

1 Introduction

Let k be a field and let $\mathrm{M}_d(k)$ be the space of $d \times d$ matrices with entries in k. In his 1961 paper [Ger61], M. Gerstenhaber proved that the unital k-algebra generated by a pair of commuting matrices $X_1, X_2 \in \mathrm{M}_d(k)$ has dimension at most d. Gerstenhaber's proof was algebro-geometric, and relied on the irreducibility of the scheme of pairs of $d \times d$ commuting matrices (a fact also proved in the earlier paper [MT55]). Linear algebraic proofs (see [BH90, LL91]) and commutative algebraic proofs (see [Wad90, Ber13]) of Gerstenhaber's theorem were later discovered.

The analog of Gerstenhaber's theorem for four or more pairwise commuting matrices is false. For example, if E_{ij} denotes the 4×4 matrix with a 1 in position

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(i, j) and 0s elsewhere, then the unital k-algebra generated by $E_{13}, E_{14}, E_{23}, E_{24}$ has k-vector space basis $I, E_{13}, E_{14}, E_{23}, E_{24}$ and thus has dimension $5 > 4$.

It is not known if the dimension of the unital k-algebra generated by three pairwise commuting matrices $X_1, X_2, X_3 \in \mathrm{M}_d(k)$ can exceed d. Determining if this three matrix analog of Gerstenhaber's theorem is true is sometimes called the *Gerstenhaber problem*. For further details and history on Gerstenhaber's theorem and the Gerstenhaber problem, see [Set11, HO15].

The Gerstenhaber problem can be viewed from a commutative-algebraic perspective, as in Proposition 1 below (see Proposition 2.4 and Corollary 2.9 of [RS18] for a proof). To state this result, we fix the following notation: given a set $X = \{X_1, \dots, X_n\} \subseteq \mathrm{M}_d(k)$ of pairwise commuting matrices, let $\mathcal{A}_X$ denote the unital k-algebra generated by the matrices in X. Let $k[x_1, \dots, x_n]$ be a polynomial ring in n variables over k and let (GP_n) denote the following statement (which is not always true):

(GP_n) Every $k[x_1, \dots, x_n]$-module N which is finite dimensional over k and which has support $\operatorname{Supp} N = (x_1, \dots, x_n)$ satisfies the inequality

$$\dim k[x_1, \dots, x_n]/\operatorname{Ann}(N) \leq \dim N.$$

Proposition 1 *Fix a positive integer n. Statement* (GP_n) *is true if and only if for every positive integer d and every set of n pairwise commuting matrices* $X = \{X_1, \dots, X_n\} \subseteq \mathrm{M}_d(k)$, *the inequality* $\dim \mathcal{A}_X \leq d$ *holds.*

Consequently, (GP_1) and (GP_2) are true, and (GP_n), $n \geq 4$, is false. Solving the Gerstenhaber problem is equivalent to determining if (GP_3) is true or false.

In this paper, we address (GP_3) in special cases. That is, we prove that the inequality

$$\dim k[x_1, x_2, x_3]/\operatorname{Ann} N \leq \dim N \tag{1}$$

holds for certain classes of modules. To motivate some of the classes that we treat, consider first the following example of a $k[x_1, x_2, x_3, x_4]$-module N for which $\dim k[x_1, x_2, x_3, x_4]/\operatorname{Ann} N > \dim N$. (This is the module associated to the standard counter-example, given above, to the four commuting matrix analog of Gerstenhaber's theorem. See [RS18, Example 1.7] for further explanation.)

Example 1 Let $S = k[x_1, x_2, x_3, x_4]$, let $\mathfrak{m} = (x_1, x_2, x_3, x_4)$, let $I = \mathfrak{m}^2 + (x_1, x_2)$, and let $J = \mathfrak{m}^2 + (x_3, x_4)$. Let $N = (S/I \times S/J)/\langle (x_3, -x_1), (x_4, -x_2) \rangle$. Note that N is 4-dimensional with basis $(1, 0), (x_3, 0), (x_4, 0), (0, 1)$. We have $\operatorname{Ann}(N) = \mathfrak{m}^2$ and so

$$5 = \dim S/\operatorname{Ann} N > \dim N = 4.$$

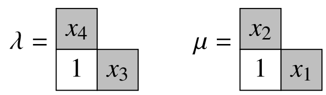
We make the following observations:

(i) N is an extension of a cyclic module by $S/\mathfrak{m}$. That is, it fits into the short exact sequence

$$0 \to S/I \xrightarrow{i} N \xrightarrow{\pi} S/\mathfrak{m} \to 0$$

such that $i(f) = (f, 0)$ and $\pi(f, g) = g$.

(ii) N is combinatorial in the sense that I and J are monomial ideals, the module N is obtained from S/I and S/J by identifying monomials in S/I with monomials in S/J, and $\operatorname{Ann} N$ is a monomial ideal.
Equivalently, N can be described in terms of two 4-dimensional analogs of Young diagrams together with "gluing" data. Indeed, with respect to the usual correspondence between monomial ideals in $k[x_1, \dots, x_n]$ and n-dimensional analogs of Young diagrams (see Section 4.1), S/I corresponds to the 4-dimensional Young diagram λ drawn below and S/J corresponds to the 4-dimensional Young diagram μ drawn below; both λ and μ are supported in 2-dimensional coordinate spaces in this case. Boxes are labelled by their corresponding monomials. Grey boxes in λ are identified with grey boxes in μ, corresponding to the relations $(x_3, 0) = (0, x_1)$ and $(x_4, 0) = (0, x_2)$ in the module N.



(2)

Then, $\dim N$ is the total number of boxes in λ plus the number of unshaded boxes in μ, and $\operatorname{Ann} N$ is the monomial ideal associated to the diagram $\lambda \cup \mu$.

Motivated by Example 1 (i), the first two listed authors of the present paper proved in [RS18, Theorem 1.5] that inequality (1) holds whenever N is an extension of a finite dimensional cyclic module $k[x_1, x_2, x_3]/I$ by a simple module $k[x_1, x_2, x_3]/(x_1, x_2, x_3)$, i.e. (GP_3) holds for such N. In this paper, the cases we consider are motivated by Example 1 (ii), and by our result in [RS18]. We briefly discuss these cases now.

Towards double extensions of cyclic modules

In light of [RS18, Theorem 1.5], it is natural to ask if (GP_3) holds for finite dimensional modules which are double extensions by $S/(x_1, x_2, x_3)$ of a cyclic module. To be precise, let $S = k[x_1, \dots, x_n]$, let $\mathfrak{m} = (x_1, \dots, x_n)$, and define a *double extension module* to be an S-module N with the following properties:

- N is finite dimensional with support $\mathfrak{m}$
- there exists an ideal I and module N_1 which fits into the following short exact sequences

$$0 \to S/I \to N_1 \to S/\mathfrak{m} \to 0, \qquad 0 \to N_1 \to N \to S/\mathfrak{m} \to 0.$$

In Section 2, we begin our study of double extension modules by proving the following:

Proposition 2 *Let N be an S module with* $\operatorname{Supp} N = \mathfrak{m}$. *If N is a double extension module satisfying* $\dim S/\operatorname{Ann} N > \dim N$, *then there exists an ideal I' and a module map*

$$\beta : (x_1^2, x_2, \dots, x_n) \to S/I'$$

satisfying $\dim I'/(I' \cap \ker \beta) > 2$.

Then, in Section 3 we prove the following:

Theorem 1 *Let $n = 3$ so that $S = k[x_1, x_2, x_3]$, and let $I \subseteq S$ be a monomial ideal with $\sqrt{I} = \mathfrak{m}$. If r is a positive integer and*

$$\beta : (x_1^r, x_2, x_3) \to S/I,$$

is a module map which maps monomials to monomials, then $\dim I/(I \cap \ker \beta) \leq r$.

In the proof of Proposition 2, we see that certain double extension modules N give rise to maps $\beta : (x_1^2, x_2, \ldots, x_n) \to S/I'$, $\sqrt{I'} = \mathfrak{m}$, and that

$$\dim S/\mathrm{Ann}\, N \leq \dim N \iff \dim I'/(I' \cap \ker(\beta)) \leq 2.$$

The following corollary is now immediate from Proposition 2 and the $r = 2$ case of Theorem 1:

Corollary 1 *Let N be a double extension $k[x_1, x_2, x_3]$-module which gives rise to a module map $\beta : (x_1^2, x_2, x_3) \to S/I'$ where I' is a finite colength monomial ideal and β maps monomials to monomials. Then N is not a counter-example to* (GP_3).

Other combinatorial classes

As pointed out in item (2) of Example 1, there are counter-examples to (GP_4) which can be described in terms of a pair of 4-dimensional Young diagrams λ and μ, together with gluing data. Indeed, in Example 1, λ and μ were glued to one another by identifying the two outer corners of λ with the two outer corners of μ. In Section 4, we investigate such 2-generated combinatorial modules. Our main result, which contrasts the four variable case, is the following (see Theorem 4 for a more precise statement):

Theorem 2 (GP_3) *holds for all modules obtained by gluing a subset of outer corners of one plane partition (a.k.a. 3-dimensional Young diagram) to another.*

Finally, in Section 5 we show that there are no counter-examples to (GP_n) of the form $N = J/I$ where $I, J \subseteq S$ are monomial ideals with $I \subseteq J$.[1]

Throughout the paper, we let $\mathbb{N}$ denote the set of non-negative integers.

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[1] While we do not know of a reference where this is proved, we would not be surprised if this result is known, perhaps with a different proof.

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2 A reformulation of the Gerstenhaber problem for double extensions of cyclic modules

By Proposition 1, the Gerstenhaber problem is true if and only if $\dim S/\operatorname{Ann} N \leq \dim N$ for all finite dimensional $k[x_1, x_2, x_3]$-modules with $\operatorname{Supp} N = (x_1, x_2, x_3)$. Clearly, if N is a finite dimensional cyclic module, so that $N = S/I$ for an ideal $I \subseteq k[x_1, x_2, x_3]$, then $\dim S/\operatorname{Ann} N \leq \dim N$. Furthermore, it was proved in [RS18] that $\dim S/\operatorname{Ann} N \leq \dim N$ for all $k[x_1, x_2, x_3]$-modules N such that N has support (x_1, x_2, x_3), and N is an extension of a cyclic module by $S/(x_1, x_2, x_3)$. In this section, and the next, we consider modules obtained from such N by further extending by $S/(x_1, x_2, x_3)$. As discussed in Section 1, we call such modules *double extension modules*. In this section, we prove Proposition 2. The ideas in the proof are similar to those used in the proofs of [RS18, Propositions 1.10 and 2.2].

Proof (Proof of Proposition 2) Let $S = k[x_1, \dots, x_n]$ and let $\mathfrak{m} = (x_1, \dots, x_n)$. Let I be a finite-colength ideal in S, let N' be an extension of S/I by $S/\mathfrak{m}$, and let N be an extension of N' by $S/\mathfrak{m}$. Furthermore, assume $\operatorname{Supp} N = \mathfrak{m}$.

The extension

$$0 \to N' \to N \to S/\mathfrak{m} \to 0$$

corresponds to a class $\alpha \in \operatorname{Ext}^1(S/\mathfrak{m}, N')$, and it was shown in the proof of [RS18, Proposition 2.2] that α lifts to a map $\alpha' : \mathfrak{m} \to N'$. It was furthermore shown in the same proof that $\dim S/\operatorname{Ann} N \leq \dim N$ if and only if

$$\dim S/\operatorname{Ann} N' + \dim \alpha'(\operatorname{Ann} N') \leq \dim N' + 1. \tag{3}$$

We now consider two cases: α' factors through the submodule S/I of N', or it does not. In the first case, α' defines a map from $\mathfrak{m}$ to S/I, and so, by [RS18, Theorem 3.1], we have $\dim \alpha'(I) \leq 1$. Furthermore, since N' is the extension of a cyclic module by $S/\mathfrak{m}$, [RS18, Theorem 1.5] implies that $\dim S/\operatorname{Ann} N' \leq \dim N'$. Adding these two inequalities together yields (3).

Consequently, if $\dim S/\operatorname{Ann} N > \dim N$, then α' does not factor through S/I. Let π be as in the bottom row of the diagram below. Then, since α' does not factor through $S/I \subseteq N'$, we see $\pi \circ \alpha'$ is surjective. We have a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & \mathfrak{m} & \longrightarrow & S/\mathfrak{m} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \simeq} & & \\
0 & \longrightarrow & S/I & \longrightarrow & N' & \xrightarrow{\pi} & S/\mathfrak{m} & \longrightarrow & 0
\end{array}$$

where $J = \ker(\pi \circ \alpha')$. The maps α' and β' define extensions

$$0 \to N' \to N \to S/\mathfrak{m} \to 0 \quad \text{and} \quad 0 \to S/I \to \widetilde{N} \to S/J \to 0,$$

respectively, and one checks that $\widetilde{N} \simeq N$.

Now, J has colength 2, so for some x_i, we have that $\{1, x_i\}$ is a basis of S/J. Without loss of generality, we may assume that $i = 1$. Consider the Lexicographic monomial order $x_n > x_{n-1} > \cdots > x_1$. Noting that $J \subseteq \mathfrak{m}$, it is easy to check that J has a Gröbner basis of the form

$$\{x_1^2 - a_1x_1, x_2 - a_2x_1, \ldots, x_n - a_nx_1\}, \quad a_j \in k.$$

Furthermore, the ideal generated by these terms has support at two distinct points unless $a_1 = 0$. Thus, $J = (x_1^2, x_2 - a_2x_1, \ldots, x_n - a_nx_1)$. Let $x_i' = x_i - a_ix_1$, $2 \le i \le n$, and let $S' = k[x_1, x_2' \ldots, x_n']$. Let $\phi : S' \to S$ be the ring isomorphism given by $x_1 \mapsto x_1$, and $x_i' \mapsto x_i - a_ix_i$, $2 \le i \le n$. Then the short exact sequence of S-modules

$$0 \to S/I \to \widetilde{N} \to S/J \to 0$$

is also a short exact sequence of S'-modules via the ring map ϕ. Furthermore, the ring isomorphism ϕ induces an isomorphism between $S/\mathrm{Ann}_S(\tilde{N})$ and $S'/\mathrm{Ann}_{S'}(\tilde{N})$. Thus, $\dim S/\mathrm{Ann}_S(\tilde{N}) \le \dim \tilde{N}$ if and only if $\dim S'/\mathrm{Ann}_{S'}(\tilde{N}) \le \dim \tilde{N}$. Re-writing each module in our new coordinates $x_1, x_2' \ldots, x_n'$ yields a short exact sequence of S'-modules of the form

$$0 \to S'/I' \to M \to S'/(x_1^2, x_2', \ldots, x_n') \to 0 \tag{4}$$

and we have $\dim S'/\mathrm{Ann}_{S'} \tilde{N} \le \dim \tilde{N}$ if and only if $\dim S'/\mathrm{Ann}_{S'} M \le \dim M$. Consequently, $\dim S/\mathrm{Ann}_S N \le \dim N$ if and only if $\dim S'/\mathrm{Ann}_{S'} M \le \dim M$.

Let $\beta : (x_1^2, x_2', \ldots, x_n') \to S'/I'$ determine the extension in (4). Then, one can check that $\mathrm{Ann}\, M = I' \cap \ker \beta$. So,

$$\begin{aligned} \dim S'/\mathrm{Ann}\, M &= \dim S'/(I' \cap \ker \beta) \\ &= \dim S'/I' + \dim I'/(I' \cap \ker \beta). \end{aligned}$$

Finally, since $\dim M = \dim S'/I' + 2$, the inequality $\dim S'/\mathrm{Ann}\, M \le \dim M$ holds if and only if $\dim I'/(I' \cap \ker \beta) \le 2$. □

We end this section with an example of the usefulness of the main idea of the above proof (which was also a key idea in [RS18]), namely, the idea to translate the statement $\dim S/\mathrm{Ann}\, N \le \dim N$ into a statement about module maps.

Example 2 ($(GP)_n$ is true for extensions of S/I by S/I) Let $S = k[x_1, \ldots, x_n]$ and consider extensions of the form

$$0 \to S/I \to N \to S/I \to 0.$$

One may check (as in the proof of [RS18, Proposition 2.2]) that the corresponding class $\alpha \in \mathrm{Ext}^1(S/I, S/I)$ is determined by a map $\beta : I \to S/I$ and that $\mathrm{Ann}\, N = I \cap \ker \beta$. So,

$$\dim S/\mathrm{Ann}\, N = \dim S/(I \cap \ker(\beta)) = \dim S/I + \dim I/(I \cap \ker(\beta)) = \dim S/I + \dim \beta(I).$$

Also, $\dim N = 2 \dim S/I$. Thus the inequality $\dim S/\mathrm{Ann}\, N \leq \dim N$ is true if and only if the inequality $\dim \beta(I) \leq \dim S/I$ is true. This latter inequality obviously holds since the codomain of β is S/I.

3 Addressing the Gerstenhaber problem for double extensions of cyclic modules in a combinatorial case

The purpose of this section is to prove Theorem 1. Throughout, let $S = k[x, y, z]$. Each ideal I will be assumed to be a finite colength monomial ideal. We say a module map $\beta : (x^r, y, z) \to S/I$ is a *monomial map* if β sends monomials to monomials. For each $\ell \geq 0$, we let

$$(S/I)_\ell := \{x^\ell y^i z^j \notin I \mid i, j \geq 0\}$$

and refer to this set of monomials as the x^ℓ*-slice of* S/I. We refer to any set of the form $(S/I)_\ell$ as an x*-slice of* S/I.

Notice that each x^ℓ-slice may be identified in a natural way with $k[y, z]/J_\ell$ where $J_\ell \subseteq k[y, z]$ is a monomial ideal. We say the x^ℓ-slice $(S/I)_\ell$ is *Gorenstein* if the socle $\mathrm{Soc}(k[y, z]/J_\ell)$ is 1-dimensional as a k-vector space; recall the socle of a $k[y, z]$-module M is the subset of elements annihilated by (y, z).

3.1 The case $I \subseteq (x^r, y, z)$

Here we assume that $I \subseteq (x^r, y, z)$ so that β restricts to a map $\beta|_I : I \to S/I$. Then $\ker(\beta|_I) = I \cap \ker(\beta)$, so

$$\beta(I) = I/(I \cap \ker(\beta)).$$

Consequently, $\dim I/(I \cap \ker(\beta)) \leq r$ if and only if $\dim \beta(I) \leq r$. We will show that this latter inequality holds. We begin by recording some properties that a map $\beta : (x^r, y, z) \to S/I$ would have to satisfy if it were a *counter-example*, that is, if $\dim \beta(I) > r$.

Our first goal is to identify those elements in I that could be mapped to nonzero elements in S/I by β. For this purpose, define

$$S_x := \{x^i \in I \mid r \le i\},$$
$$S_y := \{x^i y^j \in I \mid 0 \le i < r, j \ge 1\},$$
$$S_z := \{x^i z^l \in I \mid 0 \le i < r, l \ge 1\}.$$

Define a *border element* of S_y (respectively S_z) to be an $m \in S_y$ (respectively $m \in S_z$) such that $m/y \notin I$ (respectively $m/z \notin I$). Define a border element of S_x to be an $m \in S_x$ such that $m/x^r \notin I$. Let Ω_x, Ω_y and Ω_z be the set of border elements of S_x, S_y and S_z, respectively. Finally, define $\beta(\Omega_x), \beta(\Omega_y), \beta(\Omega_z)$ to be the submodule of S/I generated by the images, under β, of the monomials in Ω_x, Ω_y, Ω_z respectively.

Lemma 1 *If $\beta : (x^r, y, z) \to S/I$ is a monomial map then*

1. $\beta(I) \subseteq \beta(\Omega_x) + \beta(\Omega_y) + \beta(\Omega_z)$.
2. *If $x^r \mid \beta(x^r)$ (respectively $y \mid \beta(y)$, $z \mid \beta(z)$) then $\beta(\Omega_x) = 0$ (respectively $\beta(\Omega_y) = 0$, $\beta(\Omega_z) = 0$).*
3. $\dim \beta(\Omega_x) \le r$, $\dim \beta(\Omega_y) \le r$, *and* $\dim \beta(\Omega_z) \le r$.

Proof Suppose that $x^i y^j z^l \in I$. Then we must have $i \ge r$ or $j > 0$ or $l > 0$. Observe:

- If $i \ge r$ then $\beta(x^i y^j z^l) = \beta(x^r) x^{i-r} y^j z^l$, and so $y^j z^l$ divides $\beta(x^i y^j z^l)$.
- If $j > 0$ then $\beta(x^i y^j z^l) = x^i \beta(y) y^{j-1} z^l$, and so $x^i z^l$ divides $\beta(x^i y^j z^l)$.
- If $l > 0$ then $\beta(x^i y^j z^l) = x^i y^j \beta(z) z^{l-1}$, and so $x^i y^j$ divides $\beta(x^i y^j z^l)$.

Thus, if any two of the three conditions $i \ge r$, $j > 0$, $l > 0$ holds, we see that $x^i y^j z^l$ divides $\beta(x^i y^j z^l)$, and so $\beta(x^i y^j z^l) = 0$. This proves that every monomial in I which maps to a nonzero element is in one of S_x, S_y, or S_z, and so $\beta(I) \subseteq \beta(S_x) + \beta(S_y) + \beta(S_z)$.

Next, observe that the only elements of S_y that can map to non-zero elements of S/I are border elements in S_y: if $m \in S_y$ is not a border element, then $m = ym'$ for some $m' \in I$ and so $\beta(m) = \beta(y)m' = 0$. Similarly, the only elements of S_z and S_x which can map to non-zero elements of S/I are border elements. This proves (1).

Item (2) follows by noting that all elements of $\Omega_x, \Omega_y, \Omega_z$ are in I.

For item (3), one can check that there are only r distinct monomials in each of $\Omega_x, \Omega_y, \Omega_z$. For example, for each $0 \le i < r$, there is a unique j such that $x^i y^j \in \Omega_y$. □

Lemma 2 *If $\beta : (x^r, y, z) \to S/I$ is a monomial map which is a counter-example, then $x^r y, x^r z, yz \in \ker(\beta)$.*

Proof We only prove $\beta(x^r y) = 0$, the other two statements being similar. Proceed by contradiction and assume that $\beta(x^r y) \ne 0$. We have the following equality of nonzero monomials

$$x^r \beta(y) = \beta(x^r) y,$$

which implies that y divides $\beta(y)$ and x^r divides $\beta(x^r)$. Thus, $\beta(\Omega_x) = \beta(\Omega_y) = 0$ by (2) of Lemma 1. It follows that $\beta(I) \subseteq \beta(\Omega_z)$ and so $\dim \beta(I) \le r$ by (1) and (3) Lemma 1. This contradicts the fact that β is a counter-example. □

Lemma 3 *Suppose $\beta : (x^r, y, z) \to S/I$ is a monomial map which is a counter-example. Then every element of $\beta(I)$ is contained in the socle of an x-slice. Moreover, if $\ell \geq 0$ and $(S/I)_\ell$ contains a non-zero element $\omega \in \beta(\Omega_y)$, then $(S/I)_\ell$ is Gorenstein. Similarly for $\beta(\Omega_z)$.*

Proof Since $\beta(x^r)$ is killed by y and z by Lemma 2, it is clear that $\beta(\Omega_x)$ is always mapped to a socle of an x-slice.

Next, say $\omega \in \Omega_y$ maps to the x^ℓ-slice. We show $\beta(\omega)$ is in the socle of this slice. It is clear, again by Lemma 2, that $\beta(\omega)$ is in the annihilator of z. To see that $\beta(\omega)$ is also killed by y, notice that since $\omega \in \Omega_y$ we have $j > 0$. So,

$$y\beta(\omega) = \omega\beta(y) = 0$$

as $\omega \in I$. Therefore, $\beta(\omega)$ is in the socle of the x^ℓ-slice.

By symmetry in y and z, every element of $\beta(\Omega_z)$ also maps to the socle of an x-slice. Therefore, every element of $\beta(I)$ maps to the socle of an x-slice by (1) of Lemma 1.

It remains to prove that if $\omega = x^i y^j \in \Omega_y$ and $0 \neq \beta(\omega)$ is in the x^ℓ-slice, then the slice is Gorenstein. Since $\beta(\Omega_y) \neq 0$, we have that $y \nmid \beta(y)$ by (2) of Lemma 1. Thus, we may assume $\beta(y) = x^u z^v$ for some $u, v \in \mathbb{N}$. Then $\beta(\omega) = x^{i+u} y^{j-1} z^v$, so $\ell = i + u$.

Now, $\beta(x^i y) = x^\ell z^v$ is a nonzero element in the x^ℓ-slice that is killed by z, so there are no monomials in the socle of the x^ℓ-slice which have strictly smaller y-coordinate (and strictly larger z-coordinate) than $\beta(\omega)$. On the other hand, there are also no monomials in the socle of the x^ℓ-slice with strictly larger y-coordinate (and strictly smaller z-coordinate) than $\beta(\omega)$ because any monomial in the x^ℓ-slice with y-coordinate strictly larger than $\beta(\omega) = x^{i+u} y^{j-1} z^v$ would be a multiple of $\omega = x^i y^j$, which is in I. □

Corollary 2 *Let $\beta\colon (x^r, y, z) \to S/I$ be a monomial map with $I \subseteq (x^r, y, z)$. If β is a counter-example, then each x-slice contains at most one non-zero element of $\beta(I)$.*

Proof Fix an x-slice and suppose that two different monomials in I map to nonzero elements of this slice. Then, without loss of generality, our x-slice contains an element of $\beta(\Omega_y)$ as well as an element of $\beta(\Omega_x)$ or $\beta(\Omega_z)$. Since our x-slice contains an element of $\beta(\Omega_y)$, Lemma 3 tells us that our slice is Gorenstein and the $\beta(\Omega_y)$ element is in the socle. Since Lemma 3 also tells us that every element of $\beta(I)$ is in the socle of a slice, necessarily the other element of $\beta(\Omega_x)$ or $\beta(\Omega_z)$ maps to the same (unique) element of the socle. □

Lemma 4 *Let $\beta\colon (x^r, y, z) \to S/I$ be a monomial map with $I \subseteq (x^r, y, z)$ and $\beta(x^r) = 0$. Then β is not a counter-example.*

Proof We proceed by induction on r. The base case $r = 1$ is a corollary of the main theorem of [RS18]; in fact the main theorem implies the $r = 1$ case *without* the assumption that $\beta(x) = 0$.

Now suppose $r \geq 2$. Let x^d, y^e, z^f be among the minimal generators of I. Then, $r \leq d$ as $I \subseteq (x^r, y, z)$.

We claim $\beta(y^e)$ and $\beta(z^f)$ are linearly independent in S/I, otherwise we may remove the x^0-slice to get a smaller counter-example. To make this precise, first notice that y^e and z^f are the only border elements in the x^0-slice, so by Lemma 1(1), $\beta((S/I)_0) \subseteq (\beta(y^e), \beta(z^f))$ is at most one-dimensional if $\beta(y^e)$ and $\beta(z^f)$ are linearly dependent. Define a map

$$\gamma\colon K = (x^{r-1}, y, z) \to (x)/(I \cap (x)) \simeq S/(I : x)$$

by $\gamma(f) = \beta(xf)$. Since $x(I : x) = I \cap (x)$,

$$\dim \gamma((I : x)) = \dim \beta(x(I : x)) = \dim \beta(I \cap (x)).$$

Note $\gamma(x^{r-1}) = \beta(x^r) = 0$ and $(I : x) \subseteq (x^{r-1}, y, z)$. Thus by the induction hypothesis, γ is not a counter-example, so that $\dim \beta(I \cap (x)) \leq r - 1$. It follows that

$$\dim \beta(I) = \dim \beta((S/I)_0) + \dim \beta(I \cap (x)) \leq r$$

and so β is not a counter-example.

In particular, $y \nmid \beta(y)$, for otherwise $\beta(y^e) = y^{e-1}\beta(y)$ would be divisible by $y^e \in I$, hence is zero in S/I, contradicting linear independence of $\beta(y^e)$ and $\beta(z^f)$. Similarly, $z \nmid \beta(z)$.

Therefore, we may assume

$$\beta(y) = x^u z^v, \quad \beta(z) = x^s y^t, \quad \beta(x^r) = 0.$$

for some $u, v, s, t \in \mathbb{N}$. Without loss of generality, we may assume $u \leq s$.

Now we show $v = f - 1$. Since $\beta(z^f) = x^s y^t z^{f-1} \notin I$ and $\beta(yz) = x^u z^{v+1} \in I$ by Lemma 2, we must have $f - 1 < v + 1$. On the other hand, $z^f \in I$ and $\beta(y^e) = x^u y^{e-1} z^v \notin I$, so $v < f$. Thus $v = f - 1$, $\beta(y^e) = x^u y^{e-1} z^{f-1}$.

Let w be the smallest integer such that $x^{u+w} z^{f-1} = x^w \beta(y) \in I$; note that $w \leq r$ by Lemma 2. Then all nonzero elements in $\beta(\Omega_y)$ must be contained in an x^l-slice with $u \leq l < u + w$. Indeed, for $x^i y^{e_i} \in \Omega_y$ with $i \geq w$, we have $\beta(x^i y^{e_i}) = x^{i+u} y^{e_i - 1} z^{f-1} = 0$ as $x^{u+w} z^{f-1} \in I$.

Next, we consider the possible contributions from Ω_z. Let $h = x^i z^{f_i} \in \Omega_z$ be a border element, so $i \leq r$ and $f_i \leq f$.

For $i \leq u + w - 1$, we claim $\beta(h)$ is either zero or contained in an x^l-slice with $u \leq l < u + w$. By minimality of w, we see $x^{u+w-1} z^{f-1} \notin I$. Since $x^i z^{f_i} \in I$, we must have $f - 1 < f_i$; hence, $f = f_i$ and $\beta(h) = x^{s+i} y^t z^{f-1}$. Recall $x^{u+w} z^{f-1} \in I$, so if $s + i \geq u + w$, then $\beta(h) = 0$; otherwise $s + i < u + w$, i.e. $\beta(h)$ is in some x^l-slice with $u \leq l < u + w$.

Hence, only $x^i z^{f_i} \in \Omega_z$ with $u + w - 1 < i < r$ can be mapped to nonzero elements in x^l-slices with $l \geq u + w$. The number of such elements is at most $\max\{r - u - w, 0\}$.

In other words, nonzero monomials in $\beta(I) \subseteq \beta(\Omega_y) + \beta(\Omega_z)$ are either contained in an x^l-slice with $u \leq l < u + w$, or of the form $\beta(h)$ with $h = x^i z^{f_i} \in \Omega_z$ and $u + w - 1 < i < r$. It then follows from Corollary 2 that

$$\dim \beta(I) \leq w + \max\{r - u - w, 0\} \leq \max\{r - u, w\} \leq r.$$

Thus, β is not a counter-example. □

Lemma 5 *Let* $\beta\colon (x^r, y, z) \to S/I$ *be a monomial map with* $I \subseteq (x^r, y, z)$ *and*

$$\beta(\Omega_x) \subseteq \beta(\Omega_y) + \beta(\Omega_z).$$

Then β *is not a counter-example.*

Proof Consider the map $\gamma\colon (x^r, y, z) \to S/I$ with $\gamma(x^r) = 0$, $\gamma(y) = \beta(y)$, $\gamma(z) = \beta(z)$. Then $\gamma(I) = \beta(I)$, so it suffices to prove the result when $\beta(x^r) = 0$. This follows directly from Lemma 4. □

Theorem 3 *If* $\beta\colon (x^r, y, z) \to S/I$ *is a monomial map with* $I \subseteq (x^r, y, z)$*, then* β *is not a counter-example.*

Proof For contradiction suppose β is a counter-example. By Lemma 1(2) and Lemma 5, x^r does not divide $\beta(x^r)$, so $\beta(x^r) = x^a y^n z^m$ for some nonnegative integers a, n, m with $a \leq r$. Let d be minimal such that $x^d \in I$. Then $\beta(\Omega_x)$ is contained in the x^i-slices with $d - (r - a) \leq i < d$.

If $\beta(I)$ intersects an x^i-slice with $a \leq i < d - (r - a)$, then we claim that $\beta(\Omega_x) \subseteq \beta(\Omega_y) + \beta(\Omega_z)$, and so we are done by Lemma 5. To see this, we may assume without loss of generality that $\beta(\Omega_y)$ intersects the x^i-slice. Then the slice is Gorenstein and $\beta(x^{r+i-a})$ is in this slice. Moreover, $\beta(x^{r+i-a})$ is killed by y and z, i.e. it is the unique monomial in the socle of the slice, hence is contained in $\beta(\Omega_y)$. Since $\beta(\Omega_y)$ is closed under multiplication by x, it follows that $\beta(\Omega_x) \subseteq \beta(\Omega_y)$.

So, we may assume $\beta(I)$ does not intersect any x^i-slice with $a \leq i < d - (r - a)$. So, $\beta(I)$ is contained within the x^i slices for $i \in [0, a) \cup [d - (r - a), d)$. There are r such slices, so applying Corollary 2, we find $\dim \beta(I) \leq r$. □

3.2 The case $I \not\subseteq (x^r, y, z)$, and finishing the proof of Theorem 1

As above, let I be a finite colength monomial ideal and let $\beta : (x^r, y, z) \to S/I$, which sends monomials to monomials. Assume that $I \not\subseteq (x^r, y, z)$. Since I is a monomial ideal, there exists some minimal integer $m < r$ such that $x^i \in I$ for all $i \geq m$. Furthermore, our choice of m ensures that $I \subseteq (x^m, y, z)$.

Lemma 6 *In the above situation, we have* $\beta(x^m y) = 0$*, and* $\beta(x^m z) = 0$*.*

Proof This is clear since $x^m \in I$. □

Define a map $\beta' : \langle x^m, y, z\rangle \to S/I$ by $\beta'(x^m) = \beta(x^r)$, $\beta'(y) = \beta(y)$, and $\beta'(z) = \beta(z)$.

Lemma 7 β' *is a module map.*

Proof We first observe that $y\beta'(x^m) - x^m\beta'(y) = 0$. This is true since

$$y\beta'(x^m) - x^m\beta'(y) = y\beta(x^r) - x^m\beta(y) = 0 - 0 = 0.$$

Similarly, $z\beta'(x^m) - x^m\beta'(z) = 0$. Finally, $y\beta'(z) - z\beta'(y) = 0$ since $\beta'(z) = \beta(z)$ and $\beta'(y) = \beta(y)$. □

We are now ready to prove the main result of this section.

Proof (Proof of Theorem 1) If $I \subseteq (x^r, y, z)$, then we are done by Theorem 3. So, assume that $I \nsubseteq (x^r, y, z)$ and choose m to be the minimal integer $m < r$ such that $x^i \in I$ for all $i \geq m$. Let $\beta' : (x^m, y, z) \to S/I$ be as above. Then $I \subseteq (x^m, y, z)$ and by Theorem 3, we have that $\dim \beta'(I) \leq m$. Thus, by construction of β', we have

$$m \geq \dim \beta'(I) = \dim \beta(I \cap (x^r, y, z)) = \dim \frac{I \cap (x^r, y, z)}{(I \cap (x^r, y, z)) \cap \ker(\beta)} = \dim \frac{I \cap (x^r, y, z)}{I \cap \ker(\beta)}.$$

Now I is a monomial ideal, and the only monomials in I which are not in $I \cap (x^r, y, z)$ are $x^m, x^{m+1}, \ldots, x^{r-1}$. Thus, $\dim I/(I \cap (x^r, y, z)) = r - m$. This, together with the above inequality shows

$$\dim \frac{I}{I \cap \ker(\beta)} = \dim \frac{I}{I \cap (x^r, y, z)} + \dim \frac{I \cap (x^r, y, z)}{I \cap \ker(\beta)} \leq (r - m) + m = r,$$

yielding the desired result. □

4 Gluing plane partitions: addressing the Gerstenhaber problem for some 2-generated combinatorial modules

4.1 Young diagrams and skew-diagrams

Let $S = k[x_1, \ldots, x_n]$ and let I be a finite colength monomial ideal in S. Associate to S/I the set of lattice points $\mathbf{c} := (c_1, \ldots, c_n) \in \mathbb{N}^n$ such that the monomial $\mathbf{x}^\mathbf{c} := x_1^{c_1} \cdots x_n^{c_n} \in S \setminus I$. This set of lattice points is naturally identified with an n-dimensional *Young diagram* (a.k.a. *standard set* or *staircase diagram*). See [MS05, Ch. 3] for details. If K is a finite colength monomial ideal with $I \subseteq K$, associate to K/I the set of lattice points $\mathbf{c} \in \mathbb{N}^n$ such that $\mathbf{x}^\mathbf{c} \in K \setminus I$. This set of lattice points is naturally identified with $\nu := \lambda \setminus \lambda'$ where λ and λ' are the n-dimensional Young diagrams associated to S/I and S/K respectively. Observe that ν can be decomposed uniquely into $\nu_1 \cup \cdots \cup \nu_r$ such that the following hold:

1. For each ν_j and each pair of boxes $\mathbf{b}_1, \mathbf{b}_2 \in \nu_j$, there exists a sequence of moves of the form "move over one box in direction $\pm\mathbf{e}_i$" so that by starting at $\mathbf{b}_1$ and applying these moves, we end at $\mathbf{b}_2$, and we never leave ν_j in the process. Note that $\mathbf{e_i} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n$ denotes the i^{th} standard basis vector.

2. $r \in \mathbb{N}$ is minimal such that 1. holds.

We call each ν_j a *skew-diagram*, and the union $\nu = \nu_1 \cup \cdots \cup \nu_r$ the *decomposition of* ν *into skew-diagrams*. Note that each n-dimensional Young diagram is a skew-diagram.

Using the correspondence between monomials $\mathbf{x}^{\mathbf{c}}$ and their exponent vectors $\mathbf{c} \in \mathbb{N}^n$, we sometimes label a box in a skew-diagram by its coordinate $\mathbf{c} \in \mathbb{N}^n$, and sometimes by its associated monomial $\mathbf{x}^{\mathbf{c}}$. Along these lines, we say that $\mathbf{c} \in \nu$ is a *socle* of ν if $\mathbf{x}^{\mathbf{c}} \in \mathrm{Soc}(K/I)$. We let $\mathrm{Soc}(\nu)$ denote the set of socles of ν.

4.2 Background on gluing

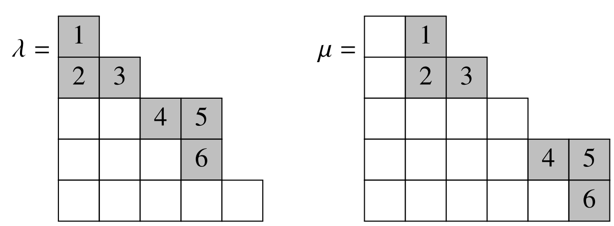
Now, let I, J, K, and L be finite colength monomial ideals in S such that $I \subseteq K$, $J \subseteq L$, and there is an S-module isomorphism $\phi : K/I \to L/J$ mapping monomials to monomials. In this section, we consider modules of the form

$$N = (S/I \times S/J)/\langle (k, -\phi(k)) \mid k \in K/I \rangle. \qquad (5)$$

Modules N from (5) have a combinatorial description, which extends the correspondence between monomial ideals in S and n-dimensional analogs of *Young diagrams*. Indeed, let λ and μ denote the n-dimensional Young diagrams associated to S/I and S/J respectively. Let ν_λ and ν_μ denote the unions of skew-diagrams associated to K/I and L/J respectively. The isomorphism $\phi : K/I \to L/J$ is a partial gluing of λ to μ by identifying the skew-diagrams in ν_λ with those in ν_μ. Note that the shapes of the skew diagrams in ν_λ agree with the shapes of those in ν_μ, otherwise ϕ would fail to be an isomorphism.

Example 3 The module N from Example 1 is of the form of (5). Here $S = k[x_1, x_2, x_3, x_4]$, $I = \mathfrak{m}^2 + (x_1, x_2)$, $J = \mathfrak{m}^2 + (x_3, x_4)$, $K = L = \mathfrak{m}$ and $\phi : K/I \to K/J$ is defined by $\phi(x_3) = x_1$ and $\phi(x_4) = x_2$. With this presentation, N corresponds to the gluing of λ and μ along the grey boxes, as depicted in (2).

Example 4 Let $S = k[x, y]$. Consider the following Young diagrams:



Observe that $I = (x^5, x^4y, x^2y^3, xy^4, y^5)$ is the monomial ideal corresponding to λ and $J = (x^6, x^4y^2, x^3y^3, x^2y^4, y^5)$ is the monomial ideal corresponding to μ. Let $\nu_\lambda \subseteq \lambda$ consist of the two shaded skew-diagrams in λ (one with the boxes labelled $1, 2, 3$ and the other with the boxes labelled $4, 5, 6$), so that ν_λ corresponds to $K/I =$

$((x^3y, x^2y^2, y^3)+I)/I$. The union of two grey skew-diagrams $\nu_\mu \subseteq \mu$ corresponds to $L/J = ((x^5, x^4y, xy^3)+J)/J$. If $\phi : K/I \to L/J$ is the map which identifies box i in ν_λ with box i in ν_μ, then we obtain a module $N = (S/I \times S/J)/\langle (k, -\phi(k)) \mid k \in K/I\rangle$.

We next we translate the inequality $\dim S/\operatorname{Ann} N \leq \dim N$ for the modules in (5) into a purely combinatorial one in terms of n-dimensional Young diagrams. This translation uses the following lemma.

Lemma 8 *Let $S = k[x_1, \ldots, x_n]$ and let $N = (S/I \times S/J)/\langle (k, -\phi(k)) \mid k \in K\rangle$ be as in* (5). *Then* $\operatorname{Ann} N = I \cap J$.

Proof Clearly $I \cap J \subseteq \operatorname{Ann} N$. On the other hand, suppose that $r \in \operatorname{Ann} N$. Then $r \cdot (1, 0) = (r, 0)$ is 0 in N and so $(r, 0)$ must be an element of the submodule $\langle (k, -\phi(k)) \mid k \in K/I\rangle \subseteq S/I \times S/J$. Since ϕ is an isomorphism, we have that $r = 0$ in S/I and thus $r \in I$. A similar argument shows that $r \in J$. □

If λ is the n-dimensional Young diagram associated to S/I and μ is the n-dimensional Young diagram associated to S/J then, by Lemma 8, we see that $S/\operatorname{Ann} N$ corresponds to the n-dimensional Young diagram $\lambda \cup \mu$. Consequently, $\dim S/\operatorname{Ann} N$ is the number of boxes in $\lambda \cup \mu$ which we denote by $|\lambda \cup \mu|$.

Example 5 In Example 1, $\lambda \cup \mu$ is the 4-dimensional Young diagram with five boxes labeled by monomials $1, x_1, x_2, x_3, x_4$.

Let N be a module determined by gluing λ to μ along ν_λ, ν_μ as explained above. Let $\nu := \nu_\lambda$. Then, $\dim N = |\lambda| + |\mu| - |\nu|$ and $\dim S/\operatorname{Ann} N = |\lambda \cup \mu|$. So, we have

$$\dim S/\operatorname{Ann} N \leq \dim N \iff |\lambda \cup \mu| \leq |\lambda| + |\mu| - |\nu| \iff |\nu| \leq |\lambda \cap \mu|. \quad (6)$$

Example 6 Continuing Example 1, we see that $|\lambda \cap \mu| = 1$, while $|\nu| = 2$. Thus, we have $|\nu| > |\lambda \cap \mu|$.

Continuing Example 4, we have $|\lambda \cap \mu| = 16$ while $|\nu| = 6$, and so $|\nu| \leq |\lambda \cap \mu|$.

Example 7 Gerstenhaber's theorem implies that for $\lambda, \mu, \nu \subseteq \mathbb{N}^2$, the inequality $|\nu| \leq |\lambda \cap \mu|$ holds. It is also not difficult to prove this directly.

Let $\nu := \nu_\lambda$ and let $\nu_1 \cup \cdots \cup \nu_r$ be the decomposition of ν into skew-diagrams (see Section 4.1). For each ν_i, let $H_0(\nu_i)$ be the height of the smallest rectangle that fits the shape ν_i. More generally, let $H_j(\nu_i)$ be the height of the smallest rectangle which fits the skew shape obtained by deleting the leftmost j columns of ν_i. We place a lexicographical order on the ν_i in ν: we say $\nu_i = \nu_j$ if ν_i and ν_j have the same shape. Otherwise, there exists some smallest $m \geq 0$ where $H_m(\nu_i) \neq H_m(\nu_j)$, in which case we say that $\nu_i > \nu_j$ if $H_m(\nu_i) > H_m(\nu_j)$. Arrange the ν_i in ν along the x-axis from largest to smallest in our order so that the largest ν_i touches both the x and y axes, and there are no columns between subsequent ν_j's, and there are no columns that contain boxes from more than one ν_i. Let η denote the smallest Young diagram which contains this configuration of ν_j's.

Now each column of λ contains boxes from at most one ν_i in ν_λ. So, we may shift all the ν_i's down to sit on the x-axis and then shift them left so that one ν_i

touches both the x and y axes, and there are no columns between subsequent ν_j's, and no columns that contain boxes from more than one ν_i. Observe that λ contains the smallest Young diagram which fits this arrangement of the ν_i, and this smallest Young diagram contains η. Thus $\eta \subseteq \lambda$. Similarly $\eta \subseteq \mu$. As η contains at least as many boxes as $|\nu|$, we have $|\nu| \leq |\lambda \cap \mu|$.

See Figure 1 for an example of the shifting processes described above.

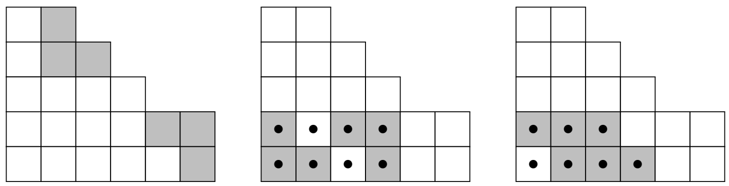


Fig. 1 Start with μ and $\nu_\mu = \nu_1 \cup \nu_2$ as in the left diagram. The grey boxes in the middle diagram are copies of ν_1, ν_2 after they have been shifted vertically down to the x-axis and then left to the origin so that ν_1 and ν_2 are next to one another with no columns in between. The boxes with bullets are the boxes in the smallest Young diagram containing all the grey boxes. The rightmost diagram consists of the ν_i ordered from largest to smallest along the x-axis, and the boxes with the bullets indicate those boxes in η.

Question 1 Does the inequality

$$|\nu| \leq |\lambda \cap \mu|$$

hold for all possible 3-dimensional λ, μ, ν as above? In other words, does the inequality $\dim S/\operatorname{Ann} N \leq \dim N$ always hold when N is a $k[x_1, x_2, x_3]$-module as in (5)?

Despite the simplicity of its 2-dimensional analog, Question 1 seems quite difficult in general. In the next section, we address it in the special case where ν is a union of corners of λ. Note that the standard four dimensional counter-example above has this form.

We end this section with two easy cases where the answer to Question 1 is "yes".

Example 8 (The case $\nu_\lambda = \nu_1$) Let λ and μ be n-dimensional Young diagrams and suppose that ν_λ contains just one skew-diagram ν_1. Then inequality (6) holds. To see this, let $\mathbf{e}_i$ denote the i-th standard basis vector and let $\nu' \subseteq \mathbb{N}^n$ denote the unique skew-diagram isomorphic to ν_λ with the property that $\nu' - \mathbf{e}_i \not\subseteq \mathbb{N}^n$ for each i. In other words, ν' is obtained from ν_λ by translating as far as possible in all $-\mathbf{e}_i$ directions. Note that $\nu' \subseteq \lambda$. Similarly, $\nu' \subseteq \mu$ and hence $\nu' \subseteq \lambda \cap \mu$, proving that $|\nu_\lambda| = |\nu'| \leq |\lambda \cap \mu|$.

Example 9 (The case where $\lambda, \mu \subseteq \mathbb{N}^3$ and each is supported in a plane) In the four variable counter-example discussed in Example 1, we saw $\lambda, \mu \subseteq \mathbb{N}^4$ were each supported in a 2-dimensional plane and $|\lambda \cap \mu| < |\nu|$. Here we see that this does not happen if $\lambda, \mu \subseteq \mathbb{N}^3$. Indeed, if λ and μ are in the same plane, we are reduced to the case of Example 7. So suppose that they are in different planes. Then each ν_i is a single box and we are in the case where we glue corners, which is proven more generally in the next section.

4.3 The Gerstenhaber problem where we glue corners

As explained in the introduction, (GP_n) is false for $n \geq 4$ due to Example 1. As further noted, this example is obtained by gluing λ to μ along corners. In contrast, we show in Theorem 4 that for $n < 4$, every 2-generated combinatorial module obtained by gluing corners does satisfy (GP_n).

We say that $(\lambda, \mu, \nu_\lambda, \nu_\mu)$ is a *counter-example* if it violates inequality (6). We say it is a *minimal counter-example* if it is a counter-example and $(\lambda', \mu', \nu_{\lambda'}, \nu_{\mu'})$ is not a counter-example whenever $\lambda' \subseteq \lambda$, $\mu' \subseteq \mu$, $\nu_{\lambda'} \subseteq \nu_\lambda$, $\nu_{\mu'} \subseteq \nu_\mu$, and $(\lambda', \mu', \nu_{\lambda'}, \nu_{\mu'}) \neq (\lambda, \mu, \nu_\lambda, \nu_\mu)$.

If additionally, each connected component of ν_λ is a singleton box, then we say $(\lambda, \mu, \nu_\lambda, \nu_\mu)$ is a *counter-example for gluing corners*, respectively a *minimal counter-example for gluing corners*.

Lemma 9 *If $(\lambda, \mu, \nu_\lambda, \nu_\mu)$ is a minimal counter-example, then*

1. $\mathrm{Soc}(\nu_\lambda) \cap (\lambda \cap \mu) = \varnothing = \mathrm{Soc}(\nu_\mu) \cap (\lambda \cap \mu)$,
2. $\mathrm{Soc}(\lambda) = \mathrm{Soc}(\nu_\lambda)$ *and* $\mathrm{Soc}(\mu) = \mathrm{Soc}(\nu_\mu)$.

Proof To prove the first assertion, assume to the contrary that $s_\lambda \in \mathrm{Soc}(\nu_\lambda) \cap (\lambda \cap \mu)$ and let $s_\mu \in \mathrm{Soc}(\nu_\mu)$ be the element to which s_λ is glued. Let $\lambda' = \lambda \setminus \{s_\lambda\}$ and $\mu' = \mu \setminus \{s_\mu\}$. Then

$$\lambda' \cap \mu' = (\lambda \cap \mu) \setminus \{s_\lambda, s_\mu\}.$$

So, $|\lambda' \cap \mu'|$ is either equal to $|\lambda \cap \mu| - 1$ or $|\lambda \cap \mu| - 2$, depending on whether s_μ is in $\lambda \cap \mu$. In either case,

$$|\lambda' \cap \mu'| \leq |\lambda \cap \mu| - 1.$$

Now, by minimality, we know $(\lambda', \mu', \nu_\lambda \setminus s_\lambda, \nu_\mu \setminus s_\mu)$ satisfies inequality (6), i.e. $|\nu_\lambda| - 1 \leq |\lambda' \cap \mu'| \leq |\lambda \cap \mu| - 1$. So, $|\nu_\lambda| \leq |\lambda \cap \mu|$, contradicting the fact that $(\lambda, \mu, \nu_\lambda, \nu_\mu)$ violates inequality (6). We have therefore shown that any minimal example must have the property that $\mathrm{Soc}(\nu_\lambda) \cap (\lambda \cap \mu) = \varnothing = \mathrm{Soc}(\nu_\mu) \cap (\lambda \cap \mu)$.

For the second assertion, if $s \in \mathrm{Soc}(\lambda) \setminus \nu_\lambda$, then let $\lambda' = \lambda \setminus s$. We have $\lambda' \cap \mu \subseteq \lambda \cap \mu$; in fact $|\lambda' \cap \mu| = |\lambda \cap \mu| - 1$ if $s \in \mu$, and $|\lambda' \cap \mu| = |\lambda \cap \mu|$ if $s \notin \mu$. Since $s \notin \nu_\lambda$, we can glue λ' to μ along ν_λ, and by minimality, we know $(\lambda', \mu, \nu_\lambda, \nu_\mu)$ is not a counter-example. So, $|\nu| \leq |\lambda' \cap \mu| \leq |\lambda \cap \mu|$, and hence $(\lambda, \mu, \nu_\lambda, \nu_\mu)$ is also not a counter-example. □

We now turn to the case of gluing corners. The following notion will play a central role.

Definition 1 We say λ is *jagged* if $|\operatorname{Soc}(\lambda \setminus s)| < |\operatorname{Soc}(\lambda)|$ for all $s \in \operatorname{Soc}(\lambda)$.

Remark 1 Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the i-th standard basis vector. Notice that λ is jagged if and only if for all $1 \leq i \leq n$ and each $s \in \operatorname{Soc}(\lambda)$, we have $s - e_i \notin \operatorname{Soc}(\lambda \setminus s)$. Equivalently, λ is jagged if and only if for each such i and s, there exists $j \neq i$ such that $s - e_i + e_j \in \lambda$.

Example 10 The standard set of $(x_1, \ldots, x_n)^m$ is jagged for every m and n. Similarly, the standard set of $(x_1, x_2)^6 + x_3(x_1, x_2)^4 + x_3^2(x_1, x_2)^3 + x_3^3(x_1, x_2)$ is jagged; notice that this is obtained by "stacking" copies of $(x_1, x_2)^{m_i}$ on top of one another. However, not every jagged λ is obtained in this manner, e.g. the standard set of $(x_1^2, x_2^2) + x_3(x_1, x_2)^2$ is also jagged.

Corollary 3 *If* $(\lambda, \mu, \nu_\lambda, \nu_\mu)$ *is a minimal counter-example for gluing corners, then*

1. $\operatorname{Soc}(\lambda) \cap (\lambda \cap \mu) = \varnothing = \operatorname{Soc}(\mu) \cap (\lambda \cap \mu)$,
2. λ *and* μ *are jagged.*

Proof The first assertion follows immediately from Lemma 9 as $\nu_\lambda = \operatorname{Soc}(\nu_\lambda)$ and $\nu_\mu = \operatorname{Soc}(\nu_\mu)$.

For the second assertion, suppose $|\operatorname{Soc}(\lambda \setminus s)| \geq |\operatorname{Soc}(\lambda)|$ for some $s \in \operatorname{Soc}(\lambda)$. Let $\lambda' = \lambda \setminus s$ and choose some $s' \in \operatorname{Soc}(\lambda') \setminus \operatorname{Soc}(\lambda)$. By Lemma 9 (2), we know $\operatorname{Soc}(\lambda) = \nu_\lambda$, so let $s_\mu \in \nu_\mu$ be the box to which s is glued. Let $\nu_{\lambda'} = (\nu_\lambda \setminus s) \cup \{s'\}$ and note that we can glue λ' to μ along $\nu_{\lambda'}$ and ν_μ; we simply glue s' to s_μ instead of gluing s to s_μ. By minimality, $(\lambda', \mu, \nu_{\lambda'}, \nu_\mu)$ is not a counter-example, so $|\nu_{\lambda'}| \leq |\lambda' \cap \mu|$. Since $s \notin \lambda \cap \mu$, we have $\lambda' \cap \mu = \lambda \cap \mu$, so

$$|\nu_\lambda| = |\nu_{\lambda'}| \leq |\lambda' \cap \mu| = |\lambda \cap \mu|,$$

which contradicts the fact that $(\lambda, \mu, \nu_\lambda, \nu_\mu)$ is a counter-example. □

We next prove a result characterizing jagged 2-dimensional Young diagrams, and proving the key property of jaggedness that we need in 3 dimensions. We introduce the following terminology.

Definition 2 Let λ be an n-dimensional Young diagram. For each $1 \leq s \leq n$ and $t \geq 0$, we let

$$\lambda^{s,t} := \{(c_1, \ldots, c_n) \in \lambda \mid c_s = t\}$$

and refer to it as the *t-th slice of λ in the x_s-direction*; it is denoted simply as λ^t when s is understood. If t is maximal such that $\lambda^{s,t} \neq \varnothing$, we refer to $\lambda^{s,t}$ as the *top slice of λ in the x_s-direction*.

Remark 2 Notice that if λ is an n-dimensional Young diagram, then each slice $\lambda^{s,t}$ can be viewed naturally as an $(n-1)$-dimensional Young diagram.

Proposition 3 *Let λ be an n-dimensional Young diagram.*

1. If $n = 2$, then λ is jagged if and only if it is the standard set of $(x_1, x_2)^k$ *for some k.*

2. If $n = 3$ and λ is jagged, then the top slice λ^t in the x_3-direction, when viewed as a 2-dimensional Young diagram, is the standard set of $(x_1, x_2)^k$ for some k.

Proof Observe that (2) follows immediately from (1) since jaggedness of λ implies jaggedness of λ^t.

We now turn to (1). It is clear that the standard set of $(x_1, x_2)^k$ is jagged. Conversely, suppose λ is jagged and let $s = x_1^a x_2^b \in \mathrm{Soc}(\lambda)$. By Remark 1, we see: (i) if $a > 0$ then $x_1^{a-1} x_2^{b+1} \in \lambda$, and (ii) if $b > 0$ then $x_1^{a+1} x_2^{b-1} \in \lambda$. Statement (i) implies that λ contains a socle in every column, i.e. for each j with $\lambda^{1,j} \neq \emptyset$, there exists k such that $x_1^j x_2^k \in \mathrm{Soc}(\lambda)$. Similarly, statement (ii) implies that λ contains a socle in every row. Together these statements imply that λ is the standard set of $(x_1, x_2)^k$ for some k. □

We can now answer Question 1 when we glue λ and μ along corners.

Theorem 4 *(GP_3) holds for 3-dimensional Young diagrams glued along corners, i.e. if the connected components of ν_λ are singleton boxes then, inequality (6) holds.*

Proof If there is a counter-example for gluing corners, then there is a minimal such counter-example $(\lambda, \mu, \nu_\lambda, \nu_\mu)$. By Corollary 3 (2), we know λ and μ are jagged. Let λ^t be the top slice of λ in the x_3-direction, and $\mu^{t'}$ the top slice of μ in the x_3-direction. Without loss of generality, $t \leq t'$.

By Proposition 3 (2), we know λ^t is of the form $(x_1, x_2)^k$ when it is viewed as 2-dimensional Young diagram. Since $\mathrm{Soc}(\lambda) \cap \mu = \emptyset$ by Corollary 3 (1), when we view the x_3-slice μ^t as a 2-dimensional Young diagram, we must have $\mu^t \subseteq (x_1, x_2)^{k-1}$. In particular, $\mu^t \subsetneq \lambda^t$.

Next, choose $x_1^a x_2^b$ in the socle of the 2-dimensional Young diagram μ^t, and let $s = x_1^a x_2^b x_3^t \in \mu$. By definition, $s + e_1, s + e_2 \notin \mu$. Since $\mu^t \subsetneq \lambda^t$ and $\mathrm{Soc}(\mu) \cap \lambda = \emptyset$ by Corollary 3 (1), we must have $s \notin \mathrm{Soc}(\mu)$. As a result, $s + e_3 \in \mu$. Let m be maximal such that $s' := s + m e_3 \in \mu$. Then $s' \in \mathrm{Soc}(\mu)$. However, this contradicts jaggedness of μ, since $s' - e_3 \in \mathrm{Soc}(\mu \setminus s')$. □

5 Addressing the Gerstenhaber problem in the monomial ideal case

Let I and J be two finite colength ideals in $S = k[x_1, \ldots, x_n]$ with $I \subseteq J$. Let $M = J/I$. Then, we have that $\mathrm{Ann}\, M = (I : J)$, and so

$$\dim S/\mathrm{Ann}\, M \leq \dim M \iff \dim S/(I : J) \leq \dim S/I - \dim S/J. \tag{7}$$

We do not know if the rightmost inequality in (7) is true in general. In this section, we show it is true for monomial ideals I and J in any number of variables. We thank Alexander Yong for the key observation that shifting overlapping n-dimensional Young diagrams appropriately can only increase the number of boxes in their intersection (see Lemma 10).

We begin with some notation. Let ν be an n-dimensional Young diagram (eg. associated to some S/I for a monomial ideal $I \subseteq S$, see Section 4.1). Given $\mathbf{a} = (a_1, \ldots, a_n)$, let $\nu_{\mathbf{a}}$ be the following shift of ν:

$$\nu_{\mathbf{a}} := \{\mathbf{c} \in \mathbb{N}^n \mid \mathbf{c} - \mathbf{a} \in \nu\}.$$

We can partition $\nu_{\mathbf{a}}$ into slices in the x_s direction. As in Definition 2, if the plane $x_s = t$ intersects $\nu_{\mathbf{a}}$ non-trivially, we define the t-slice of $\nu_{\mathbf{a}}$ to be the set of all $\mathbf{c} \in \nu_{\mathbf{a}}$ such that $c_s = t$. We refer to the $t = a_s$ slice as the *bottom slice* of $\nu_{\mathbf{a}}$ in the x_s direction. Let $\mathbf{e}_s$ be the s^{th} standard basis vector, and note that if $\mathbf{c} \in \nu_{\mathbf{a}}$ is not in the bottom slice in the x_s direction, then $\mathbf{c} - \mathbf{e}_s$ is still an element of $\nu_{\mathbf{a}}$.

Lemma 10 *Let $\nu^1, \ldots, \nu^r$ be n-dimensional Young diagrams, and let $\boldsymbol{a}(1), \ldots, \boldsymbol{a}(r) \in \mathbb{N}^n$. Fix some $1 \le s \le n$ and assume that*

$$a(1)_s = a(2)_s = \cdots = a(l)_s > a(l+1)_s \ge \cdots \ge a(r)_s, \tag{8}$$

for some $1 \le l \le r$. Then,

$$\left| \bigcup_{i=1}^{r} \nu^i_{\boldsymbol{a}(i)} \right| \ge \left| \bigcup_{i=1}^{l} \nu^i_{\boldsymbol{a}(i)-\mathbf{e}_s} \cup \bigcup_{i=l+1}^{r} \nu^i_{\boldsymbol{a}(i)} \right|.$$

Proof Let $\nu^{(1)} = \bigcup_{i=1}^{l} \nu^i_{\mathbf{a}(i)}$, $\nu^{(2)} = \bigcup_{i=l+1}^{r} \nu^i_{\mathbf{a}(i)}$ and $\nu^{(1)} - \mathbf{e}_s = \bigcup_{i=1}^{l} \nu^i_{\mathbf{a}(i)-\mathbf{e}_s}$. Then $|\nu^{(1)}| = |\nu^{(1)} - \mathbf{e}_s|$ since $\nu^{(1)} - \mathbf{e}_s$ is just a shift of $\nu^{(1)}$ in the $-\mathbf{e}_s$ direction. So, to prove the lemma, it suffices to show that $|\nu^{(1)} \cap \nu^{(2)}| \le |(\nu^{(1)} - \mathbf{e}_s) \cap \nu^{(2)}|$. To do this, we will show that for each $\mathbf{b} \in \nu^{(1)} \cap \nu^{(2)}$, we have $\mathbf{b} - \mathbf{e}_s \in (\nu^{(1)} - \mathbf{e}_s) \cap \nu^{(2)}$.

If $\mathbf{b} \in \nu^{(1)} \cap \nu^{(2)}$ then $\mathbf{b}$ is simultaneously in $\nu^i_{\mathbf{a}(i)}$, for some $1 \le i \le l$, and in $\nu^j_{\mathbf{a}(j)}$, for some $l+1 \le j \le r$. Then, it is clear by definition that $\mathbf{b} - \mathbf{e}_s$ is in $\nu^i_{\mathbf{a}(i)-\mathbf{e}_s}$. To see that $\mathbf{b} - \mathbf{e}_s \in \nu^j_{\mathbf{a}(j)}$, recall that $a(i)_s > a(j)_s$ by the assumption (8). Thus $\mathbf{b}$ is not in the bottom slice of $\nu^j_{\mathbf{a}(j)}$ in the x_s direction. Hence, $\mathbf{b} - \mathbf{e}_s$ is still in $\nu^j_{\mathbf{a}(j)}$ as noted above the statement of the present lemma. □

Proposition 4 *Let I and J be monomial ideals in $k[x_1, \ldots, x_n]$ with $I \subseteq J$. Then (GP_n) is true for J/I.*

Proof We first prove the following general combinatorial statement: if $\nu^1, \ldots, \nu^r$ are n-dimensional Young diagrams and $\mathbf{a}(1), \ldots, \mathbf{a}(r) \in \mathbb{N}^n$, then

$$\left| \bigcup_{i=1}^{r} \nu^i \right| \le \left| \bigcup_{i=1}^{r} \nu^i_{\mathbf{a}(i)} \right|. \tag{9}$$

We proceed by induction on the maximum distance of a vector $\mathbf{a}(i)$ to a coordinate hyperplane. More precisely, we induct on

$$\max\{t \in \mathbb{N} \mid \exists s \in [n] \text{ and } i \in [r] \text{ such that } \mathbf{a}(i)_s = t\}.$$

If $t = 0$, then $\nu^i_{\mathbf{a}(i)} = \nu^i$ for all i, and so (9) holds trivially. So, suppose $t > 0$, and choose any s, i such that $\mathbf{a}(i)_s = t$. After possibly re-labelling we may assume $t = \mathbf{a}(1)_s \geq \mathbf{a}(2)_s \geq \cdots \geq \mathbf{a}(r)_s$. If all of these inequalities are equalities, then define $\mathbf{a}'(i) = \mathbf{a}(i) - \mathbf{e}_s$ for each $1 \leq i \leq r$. Observe that (9) holds if and only if it holds upon replacing each $\mathbf{a}(i)$ by $\mathbf{a}'(i)$, as $\bigcup_{i=1}^r \nu^i_{\mathbf{a}'(i)}$ is just a shift of $\bigcup_{i=1}^r \nu^i_{\mathbf{a}(i)}$ backwards by one unit in the x_s direction.

If not all inequalities are equality then there is a first occurrence of a strict inequality $\mathbf{a}(l)_s > \mathbf{a}(l+1)_s$ at some point in the chain. In this case, define $\mathbf{a}'(i) = \mathbf{a}(i) - \mathbf{e}_s$, for $1 \leq i \leq l$, and $\mathbf{a}'(i) = \mathbf{a}(i)$, for $l+1 \leq i \leq r$. Then, Lemma 10 implies that (9) holds if it holds upon replacing each $\mathbf{a}(i)$ by $\mathbf{a}'(i)$.

In either of the above two cases, the maximum distance t' of an $\mathbf{a}'(i)$ to a coordinate hyperplane is still at most t. If it happens that $t' < t$, then the induction hypothesis yields the desired result. If $t' = t$, we can repeat the above process of shifting the various $\nu^i_{\mathbf{a}'(i)}$ until the maximum distance to a coordinate hyperplane does drop. It eventually will drop since there are only finitely many coordinate directions in which to shift. Hence (9) holds by induction.

The statement of the Proposition now follows: let $J = (x^{\mathbf{a}(1)}, \ldots, x^{\mathbf{a}(r)})$. Let $\nu_J := \{\mathbf{c} \in \mathbb{N}^n \mid \mathbf{x}^{\mathbf{c}} \text{ is nonzero in } J/I\}$ and observe that $\nu_J = \bigcup_{i=1}^r \nu^i_{\mathbf{a}(i)}$, where

$$\nu^i_{\mathbf{a}(i)} = \{\mathbf{c} \in \mathbb{N}^n \mid \mathbf{x}^{\mathbf{c}} \text{ is nonzero in } ((\mathbf{x}^{\mathbf{a}(i)}) + I)/I\},$$

and ν^i is the shift of $\nu^i_{\mathbf{a}(i)}$ to the origin, that is, $\nu^i = \{\mathbf{c} - \mathbf{a}(i) \mid \mathbf{c} \in \nu^i_{\mathbf{a}(i)}\}$. Let $\widetilde{\nu} := \bigcup_{i=1}^r \nu^i$. Then, $\dim(J/I) = |\nu_J|$ and $\dim S/\mathrm{Ann}(J/I) = \dim S/(I : J) = |\widetilde{\nu}|$. The above induction argument implies that $|\widetilde{\nu}| \leq |\nu_J|$ as desired. □

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