

A "Bottom Up" Characterization of Smooth Deligne–Mumford Stacks

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We prove a structure theorem for relative coarse space morphisms from smooth Deligne–Mumford stacks, showing that each such map can be decomposed in terms of root stack and canonical stack morphisms. We explain how our result can be understood locally in terms of pseudo-reflections. Lastly, we give an application to equivariant K -theory, give several examples illustrating our result, and pose some related questions.

1 Introduction

The main result of this article shows that if \mathcal{X} is a smooth Deligne–Mumford stack, then every relative coarse space map $\mathcal{X} \rightarrow \mathcal{Y}$ can be expressed in terms of two simple procedures applied iteratively: canonical stack constructions and root stack constructions. We state the main theorem after briefly recalling these two stack-theoretic constructions.

If k is a field, then a k -scheme U is said to have *tame quotient singularities* if it is *étale* locally a quotient of a smooth variety by a finite group of order relatively prime to the characteristic of k . Vistoli showed in [17] that for any such scheme, there is a canonical smooth tame Deligne–Mumford stack U^{can} and coarse space morphism

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$U^{\text{can}} \rightarrow U$ which is an isomorphism over the smooth locus U^{sm} . We say a Deligne–Mumford stack \mathcal{Y} has tame quotient singularities if there is an *étale* cover $U \rightarrow \mathcal{Y}$, where U is a scheme with tame quotient singularities. As with schemes, there is a canonical smooth Deligne–Mumford stack \mathcal{Y}^{can} and relative coarse space map $\mathcal{Y}^{\text{can}} \rightarrow \mathcal{Y}$. Such a map is called a *canonical stack morphism*, and satisfies a universal property described in the next section.

The second stack-theoretic construction we make use of is *root stack morphisms*, which were introduced independently in [2, 5]. Given an effective Cartier divisor \mathcal{C} on a stack \mathcal{Z} and a positive integer n , the root stack $\sqrt[n]{\mathcal{C}/\mathcal{Z}} \rightarrow \mathcal{Z}$ is universal among morphisms for which \mathcal{C} pulls back to n times a Cartier divisor.

Theorem 1. Let $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ be a relative coarse space map, where \mathcal{X} is a smooth separated tame Deligne–Mumford stack of finite type over a field k with trivial generic stabilizer. Let $\mathcal{D} \subseteq \mathcal{Y}$ be the ramification divisor of π , with irreducible components \mathcal{D}_i and ramification degrees e_i . Let $\sqrt{\mathcal{D}/\mathcal{Y}^{\text{can}}}$ denote the root stack of \mathcal{Y}^{can} with order e_i along \mathcal{D}_i . (Since $\mathcal{Y}^{\text{can}} \rightarrow \mathcal{Y}$ is an isomorphism away from codimension 2, it induces a bijection between divisors on \mathcal{Y} and divisors on \mathcal{Y}^{can} . We are slightly abusing notation by conflating the two.)

Then $\sqrt{\mathcal{D}/\mathcal{Y}^{\text{can}}}$ has tame quotient singularities and π factors as follows:

$$\mathcal{X} \cong \sqrt{\mathcal{D}/\mathcal{Y}^{\text{can}}}^{\text{can}} \rightarrow \sqrt{\mathcal{D}/\mathcal{Y}^{\text{can}}} \rightarrow \mathcal{Y}^{\text{can}} \rightarrow \mathcal{Y}.$$

Moreover, if \mathcal{D} is a Cartier divisor on \mathcal{Y} (so that $\sqrt{\mathcal{D}/\mathcal{Y}}$ is defined), then $\sqrt{\mathcal{D}/\mathcal{Y}}$ has tame quotient singularities and π factors as follows:

$$\mathcal{X} \cong \sqrt{\mathcal{D}/\mathcal{Y}}^{\text{can}} \rightarrow \sqrt{\mathcal{D}/\mathcal{Y}} \rightarrow \mathcal{Y}. \quad \square$$

Remark 2. For \mathcal{X} with non-trivial generic stabilizer, Theorem 1 can be combined with [3, Theorem A.1], which shows that \mathcal{X} is a Gerbe over a smooth stack with trivial generic stabilizer. □

Remark 3. In the statement of Theorem 1, there are several ways one can construct $\sqrt{\mathcal{D}/\mathcal{Y}^{\text{can}}}$. For example, it can be constructed as an iterated sequence of root stacks along the irreducible components $e_1\sqrt{\mathcal{D}_1/\mathcal{Y}^{\text{can}}}/\dots/e_m\sqrt{\mathcal{D}_m/\mathcal{Y}^{\text{can}}}$. Alternatively, if say $e_1 = e_2$, one can combine the two steps of rooting along \mathcal{D}_1 and \mathcal{D}_2 into a single step $e_1\sqrt{(\mathcal{D}_1 \cup \mathcal{D}_2)/\mathcal{Y}^{\text{can}}}$. As noted in the last paragraph of [10, Section 1.3b], these two stacks are *not* isomorphic. However, the proof of Theorem 1 shows that both have tame quotient singularities and their canonical stacks are isomorphic. □

Remark 4. The proof we present can be applied if k is an arbitrary regular excellent base, but we work over a field for clarity. See [16] for the construction of canonical stacks in this case. The remainder of the proof applies directly. \square

For every point y of a separated Deligne–Mumford stack \mathcal{Y} , there is a linear action of a finite group G acting on a vector space V and an isomorphism between an *étale* neighborhood of $y \in \mathcal{Y}$ and an *étale* neighborhood of $0 \in [V/G]$. It is therefore reasonable to ask for a local description of Theorem 1 in terms of the representation theory of the G -action on V .

Theorem 5 (Local description of Theorem 1). Suppose $\mathcal{X} = [V/G]$ and $\mathcal{Y} = V/G$, where V is a vector space over k on which an abstract finite group G acts linearly and faithfully. Let $H \subseteq G$ be the subgroup generated by pseudoreflections of V , and $H' \subseteq H$ be its commutator subgroup. Then the factorization of the coarse space morphism in Theorem 1 is

$$\begin{array}{ccccccc}
 \mathcal{X} \cong \sqrt{\mathcal{D}/\mathcal{Y}^{\text{can}^{\text{can}}}} & \longrightarrow & \sqrt{\mathcal{D}/\mathcal{Y}^{\text{can}}} & \longrightarrow & \mathcal{Y}^{\text{can}} & \longrightarrow & \mathcal{Y} \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 [V/G] & \longrightarrow & [(V/H')/(G/H')] & \longrightarrow & [(V/H)/(G/H)] & \longrightarrow & V/G.
 \end{array}$$

Remark 6. The proof of Theorem 5 shows that for a quotient stack $\mathcal{X} = [U/G]$, the ramification divisor of the map $[U/G] \rightarrow U/G$ is the image of the pseudoreflection divisors, and the ramification degrees are the orders of the pseudoreflections. \square

As a consequence of Theorem 1, we see that if \mathcal{X} is a smooth separated tame Deligne–Mumford stack of finite type over k with trivial generic stabilizer, it can be recovered from its coarse space X and the *orders* of the stabilizers of codimension 1 points. The exact group structures of stabilizers of codimension 1 points, as well as the stabilizer groups of higher codimension points, are then completely determined. We show in Example 1 that this characterization of DM stacks does not hold if we drop the smoothness or separateness hypotheses.

This bottom-up perspective has already seen a couple of applications. It played a central role in the authors’ work addressing a question posed by William Fulton [11] concerning whether or not a variety with quotient singularities can be globally written as the quotient of a smooth variety by a finite group. It has also been used as motivation for a conjecture in Gromov–Witten theory [15, 1.1].

Finally, we mention that our result has the following additional application in equivariant K -theory. In [8, Theorem 5.1], Dhillon and Kobyzev compute $K_G(X)$ as a direct sum of $K(X)$ and certain pieces coming from the ramification divisors of the coarse space map $[X/G] \rightarrow X/G$; here X is smooth and G is a finite group. Their result relies on having a description of $[X/G]$ in terms of its coarse space, which they obtain under certain assumptions [8, 4.5]. Since Theorem 1 gives such a description unconditionally, we obtain a mild generalization of their result.

Corollary 7. The isomorphism of [8, Theorem 5.1] computing equivariant K -theory $K_G(X)$ holds under weaker hypotheses: it is not necessary to assume that the inertia is generated in codimension 1 or that the stabilizers are abelian. \square

2 Background and preliminary results

We begin by recalling the definition of the ramification locus, branch locus, and ramification indices. Let \mathcal{X} and \mathcal{Y} be locally Noetherian Deligne–Mumford stacks and $f: \mathcal{X} \rightarrow \mathcal{Y}$ a morphism that is locally of finite presentation. Since $\Omega_{\mathcal{X}/\mathcal{Y}}^1$ is coherent, its support is a closed substack. We define the *ramification locus* of f to be this closed substack with its reduced structure and define the *branch locus* of f to be the image of the ramification locus with its reduced structure. If f takes codimension 1 points to codimension 1 points (as is the case for relative coarse space maps), and \mathcal{D} is a divisorial component of the ramification locus, then we define the ramification index $e_{\mathcal{D}}$ as follows. Choose an *étale* cover $U \rightarrow \mathcal{Y}$ and an *étale* cover V of $U \times_{\mathcal{Y}} \mathcal{X}$, let $v \in V$ be a point mapping to the generic point of \mathcal{D} , and let u be its image under the map $V \rightarrow U$. Then we define $e_{\mathcal{D}}$ to be the ramification index $e_{v/u}$ of v over u , that is, $\mathcal{O}_{U,u} \rightarrow \mathcal{O}_{V,v}$ is a local morphism of discrete valuation rings and we define $e_{\mathcal{D}}$ to be the integer e such that $\mathfrak{m}_u \mathcal{O}_{V,v} = \mathfrak{m}_v^e$ where \mathfrak{m}_u and \mathfrak{m}_v are the maximal ideals of $\mathcal{O}_{U,u}$ and $\mathcal{O}_{V,v}$, respectively. The ramification index $e_{\mathcal{D}}$ is well-defined since any two choices of (U, V, v, u) and (U', V', v', u') can be refined to a common covering; hence we can assume there is a commutative diagram

$$\begin{array}{ccc} V' & \xrightarrow{p} & V \\ \downarrow & & \downarrow \\ U' & \xrightarrow{q} & U \end{array}$$

where p and q are *étale* coverings, $p(v') = v$, and $q(u') = u$. Since ramification indices multiply in towers and *étale* maps have trivial ramification index, we see $e_{v'/u'} = e_{v'/u}e_{u'/u} = e_{v'/u} = e_{v'/v}e_{v/u} = e_{v/u}$.

The first stack-theoretic construction we employ is that of canonical stacks. If X is a finite type algebraic space, then [17, 2.9 and the proof of 2.8] shows X has tame quotient singularities if and only if X is the coarse space of a smooth tame Deligne–Mumford stack. Moreover, in this case there is a smooth tame Deligne–Mumford stack X^{can} with coarse space X such that the coarse space morphism $X^{\text{can}} \rightarrow X$ is an isomorphism away from codimension 2. We refer to X^{can} as *the canonical stack over X* .

Remark 1 (Universal property of canonical stacks [10, Theorem 4.6]). Suppose \mathcal{X} is a smooth Deligne–Mumford stack with coarse space morphism $\pi : \mathcal{X} \rightarrow X$ which is an isomorphism away from a locus of codimension at least 2. Then \mathcal{X} is universal (terminal) among smooth Deligne–Mumford stacks with trivial generic stabilizer and a dominant codimension-preserving morphism to X . □

This universal property is stable under base change by *étale* morphisms (or any other codimension-preserving morphisms). That is, for $U \rightarrow X$ *étale*, $X^{\text{can}} \times_X U$ is the canonical stack over U . By descent, every algebraic stack \mathcal{Y} with tame quotient singularities has a canonical stack \mathcal{Y}^{can} and a coarse space morphism $\pi : \mathcal{Y}^{\text{can}} \rightarrow \mathcal{Y}$ which is an isomorphism away from a set of codimension at least 2.

The second stack-theoretic construction we employ is that of root stacks (see [10, Section 1.3] for a brief introduction). Given an effective Cartier divisor \mathcal{C} on a stack \mathcal{Z} and a positive integer n , the n th root of \mathcal{C} , denoted $\sqrt[n]{\mathcal{C}/\mathcal{Z}}$, is the \mathcal{Z} -stack which is universal (terminal) among all \mathcal{Z} -stacks for which \mathcal{C} pulls back to n times another divisor. Explicitly, the root stack is given by the Cartesian diagram below. Recall that \mathcal{C} defines a morphism $\mathcal{Z} \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ since $[\mathbb{A}^1/\mathbb{G}_m]$ parameterizes line bundles with sections (i.e., effective Cartier divisors).

$$\begin{array}{ccc} \sqrt[n]{\mathcal{C}/\mathcal{Z}} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow \hat{n} \\ \mathcal{Z} & \xrightarrow{c} & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

Here, the n th power map $\hat{n} : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ takes a line bundle with section (\mathcal{L}, s) to $(\mathcal{L}^{\otimes n}, s^{\otimes n})$.

Lemma 2. Root stack morphisms and canonical stack morphisms have affine diagonal. \square

Proof. To check that root stack morphisms have affine diagonal, it is enough to show that the n th power map $[\mathbb{A}^1/\mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ has affine diagonal. Since this can be checked smooth locally on the target, and $[\mathbb{A}^1/\mu_n] = \mathbb{A}^1 \times_{[\mathbb{A}^1/\mathbb{G}_m]} [\mathbb{A}^1/\mathbb{G}_m]$, the result follows.

We next consider canonical stack morphisms. If H is a finite group acting on an affine variety U , then the coarse space map $[U/H] \rightarrow U/H$ has affine diagonal. Since all canonical stack morphisms have this form smooth locally on the target, they have affine diagonal as well. \blacksquare

Lemma 3. Suppose $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of algebraic stacks, and $\mathcal{Z} \rightarrow \mathcal{Y}$ is a surjective locally finite type morphism. Then f is representable if and only if $f_{\mathcal{Z}}: \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$ is representable. \square

Proof. By [7, Corollary 2.2.7], a morphism is representable if and only if its geometric fibers are algebraic spaces. Since $\mathcal{Z} \rightarrow \mathcal{Y}$ is locally finite type, the geometric fibers of f and $f_{\mathcal{Z}}$ are the same. \blacksquare

Lemma 4. Let \mathcal{U} be a separated finite type Deligne–Mumford stack such that every irreducible component of \mathcal{U} has trivial generic stabilizer. If \mathcal{U} is étale over its coarse space, then \mathcal{U} is an algebraic space. \square

Proof. It suffices to look *étale* locally on the coarse space of \mathcal{U} . We can therefore assume $\mathcal{U} = [V/K]$, where V is an algebraic space and K is the stabilizer of a geometric point v of V by [1, Lemma 2.2.3 and its proof]. (In the proof of [1, Lemma 2.2.3], the group Γ is the stabilizer of the x_0 .) Then the composition $V \rightarrow [V/K] \rightarrow V/K$ is *étale*, so K acts trivially on every jet space of v by the formal criterion for *étaleness*. On the other hand, K acts faithfully on some jet space of v by the proof of [9, Proposition 4.4]. (The smoothness assumption in their result is not needed, and their hypothesis that no non-identity component of the stabilizer group scheme dominate is equivalent to our hypothesis that every irreducible component of \mathcal{U} have trivial generic stabilizer.) Thus, K is trivial, so $\mathcal{U} = V$ is an algebraic space. \blacksquare

Corollary 5. Let \mathcal{U} and \mathcal{Y} be smooth separated tame Deligne–Mumford stacks. Suppose $g: \mathcal{U} \rightarrow \mathcal{Y}$ is a birational quasi-finite morphism, and the ramification locus in \mathcal{U} is of codimension >1 . Then g is representable and étale. \square

Proof. By Lemma 3, we can replace \mathcal{Y} by an *étale* cover, and so can assume $\mathcal{Y} = Y$ is a scheme. Then \mathcal{U} has trivial generic stabilizer and g factors through the coarse space U of \mathcal{U} .

We show that the induced map $\bar{g}: U \rightarrow Y$ is unramified in codimension 1. Let $u \in U$ be a point of codimension 1 and let $y \in Y$ be its image in Y . Being the coarse space of a smooth tame Deligne–Mumford stack, U has tame quotient singularities. In particular, it is normal, and so $\mathcal{O}_{U,u}$ is a discrete valuation ring. If $V \rightarrow U$ is an *étale* cover and $v \in V$ is a point of codimension 1 which maps to u , then $\mathcal{O}_{V,v}$ is unramified over $\mathcal{O}_{Y,y}$ by hypothesis. Since ramification indices multiply in towers of discrete valuation rings, we see that $\mathcal{O}_{U,u}$ is unramified over $\mathcal{O}_{Y,y}$ as well.

Now we have that U is normal, Y is smooth, and \bar{g} is dominant and unramified in codimension 1, so the purity of the branch locus theorem [14, Exposé X, Theorem 3.1] shows that \bar{g} is *étale*. Similarly, $V \rightarrow Y$ is a map between two smooth varieties which is dominant and unramified in codimension 1, so it is *étale* by purity. It follows that g is *étale*. Since g and \bar{g} are *étale*, the coarse space map $\mathcal{U} \rightarrow U$ is *étale*, and hence an isomorphism by Lemma 4. This proves representability of g . ■

Corollary 6. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of smooth separated tame Deligne–Mumford stacks with trivial generic stabilizers. If f is unramified in codimension 1 and induces an isomorphism of coarse spaces, then f is an isomorphism. □

Proof. Let $\pi: \mathcal{Y} \rightarrow Y$ be the coarse space of \mathcal{Y} . By assumption, the composition $\pi \circ f: \mathcal{X} \rightarrow Y$ is a coarse space morphism. Since \mathcal{X} and \mathcal{Y} have trivial generic stabilizers, these coarse space morphisms are birational, so f is birational. Since \mathcal{X} and \mathcal{Y} are proper and quasi-finite over Y [6, Theorem 1.1], f is proper and quasi-finite. Since f is unramified in codimension 1, it is representable by Corollary 5. Zariski’s Main Theorem [10, Theorem C.1] then shows that f is an isomorphism. ■

3 Proof of Theorem 1

Replacing \mathcal{Y} be an *étale* cover, we immediately reduce to the case where $\mathcal{Y} = X$ is a scheme. By the universal property of canonical stacks (Remark 1), the coarse space morphism $\pi: \mathcal{X} \rightarrow X$ factors through the canonical stack. Since $X^{\text{can}} \rightarrow X$ is an isomorphism away from codimension 2, we have that \mathcal{D} is the ramification divisor of the map $\tilde{\pi}: \mathcal{X} \rightarrow X^{\text{can}}$.

Let \mathcal{D}_i be the irreducible components of \mathcal{D} and suppose $\tilde{\pi}$ is ramified over \mathcal{D}_i with order e_i . Let $\tilde{\mathcal{X}} = \sqrt{\mathcal{D}/X^{\text{can}}}$ be obtained from X^{can} by taking the e_i th root along

\mathcal{D}_i for all i . By the universal property of root stacks, we have an induced morphism $g: \mathcal{X} \rightarrow \tilde{\mathcal{X}}$.

Each e_i is relatively prime to the characteristic of k ; indeed, this can be checked *étale* locally on X , so we may assume \mathcal{X} is a quotient of a smooth scheme by a finite group G which stabilizes a point [1, Lemma 2.2.3 and its proof]. The ramification orders are then orders of subgroups of G , which are relatively prime to the characteristic of k since \mathcal{X} is assumed to be tame. Since the e_i are prime to the characteristic of k , $\tilde{\mathcal{X}}$ is a tame Deligne–Mumford stack. Since $\tilde{\mathcal{X}} \rightarrow X^{\text{can}}$ ramifies with order e_i over \mathcal{D}_i and ramification orders multiply in towers of discrete valuation rings, we have that g is unramified in codimension 1.

Let $\mathcal{U} \subseteq X^{\text{can}}$ be the complement of the singular loci of the \mathcal{D}_i and the intersections of the \mathcal{D}_i . The restriction of \mathcal{D} to \mathcal{U} is a smooth divisor, so the open substack $\tilde{\mathcal{X}} \times_{X^{\text{can}}} \mathcal{U}$ of $\tilde{\mathcal{X}}$ is a smooth Deligne–Mumford stack by [10, 1.3.b(3)]. By Corollary 6, g restricts to an isomorphism of open substacks $\mathcal{X} \times_{X^{\text{can}}} \mathcal{U} \rightarrow \tilde{\mathcal{X}} \times_{X^{\text{can}}} \mathcal{U}$. These open substacks have complements of codimension at least 2, so g is Stein (i.e., $g_* \mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\tilde{\mathcal{X}}}$).

By Lemma 2, $\tilde{\mathcal{X}} \rightarrow X$ has affine diagonal. As $\pi: \mathcal{X} \rightarrow X$ is a coarse space morphism of a tame stack, it is cohomologically affine [3, Theorem 3.2]. Since \mathcal{X} is separated, X is separated [6, Theorem 1.1], and so X has quasi-affine diagonal. By [4, Proposition 3.13], $g: \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ is cohomologically affine. As g is Stein and cohomologically affine, it is a good moduli space morphism, so it is universal for maps to algebraic spaces [4, Theorem 6.6], so it is a relative coarse space morphism. (Technically, this is not a direct application of [4, Theorem 6.6], as Alper’s result assumes $\tilde{\mathcal{X}}$ is an algebraic space. However, since algebraic spaces are sheaves in the smooth topology, this property may be checked smooth locally on $\tilde{\mathcal{X}}$. Good moduli space morphisms are stable under base change by [4, Proposition 4.7(i)], so we are reduced to the case when $\tilde{\mathcal{X}}$ is an algebraic space.) Since \mathcal{X} is smooth and tame, it follows that $\tilde{\mathcal{X}}$ has tame quotient singularities. Since g is an isomorphism away from codimension 2, it is a canonical stack morphism by Remark 1. This completes the proof that π factors as $\mathcal{X} \cong \sqrt{\mathcal{D}/X^{\text{can}}}^{\text{can}} \rightarrow \sqrt{\mathcal{D}/X^{\text{can}}} \rightarrow X^{\text{can}} \rightarrow X$.

If $\mathcal{D} \subseteq \mathcal{Y} = X$ is Cartier, then this proof may be modified by replacing X^{can} by X and removing the singular locus of X from \mathcal{U} . The modified proof shows that π factors as $\mathcal{X} \cong \sqrt{\mathcal{D}/X}^{\text{can}} \rightarrow \sqrt{\mathcal{D}/X} \rightarrow X$.

4 Proof of Theorem 5

For a finite group K acting linearly on a vector space U , the coarse space morphism $[U/K] \rightarrow U/K$ is a canonical stack morphism if and only if K acts without pseudoreflections.

By the Chevalley–Shephard–Todd theorem, V/H is a vector space. Moreover, G/H acts linearly on V/H without pseudoreflections. Therefore $[(V/H)/(G/H)] \rightarrow V/G$ is a canonical stack morphism. This is the usual construction of the canonical stack.

The action of H' on V has no pseudoreflections since commutators act with determinant 1, and pseudoreflections do not. Therefore $[V/H'] \rightarrow V/H'$ is a canonical stack morphism. Since $(V/H') \rightarrow [(V/H')/(G/H')]$ is an *étale* cover, and the property of being a canonical stack morphism can be checked *étale* locally on the target, $[V/G] \rightarrow [(V/H')/(G/H')]$ is also a canonical stack morphism.

It remains to show that $[(V/H')/(G/H')] \rightarrow [(V/H)/(G/H)]$ is a root stack morphism. Since this is a property which can be checked *étale* locally on the target, it suffices to show that $[(V/H')/(H/H')] \rightarrow V/H$ is a root stack morphism.

For any pseudoreflection hyperplane W , let $H \cdot W$ be the set of H -translates of W . For each hyperplane U in this orbit, choose a linear function ℓ_U which vanishes on U , and let $f_{H \cdot W} = \prod_{U \in H \cdot W} \ell_U$. The vanishing locus of $f_{H \cdot W}$ is H -invariant, so H acts on $f_{H \cdot W}$ by some character $\chi_{H \cdot W}$ of H .

Given a pseudoreflection $r \in H$, we can break $H \cdot W$ up into r -orbits. We analyze the contribution of each of these cycles to $\chi_{H \cdot W}(r)$. If a hyperplane U in $H \cdot W$ is fixed by r , then either r is a pseudoreflection through U (in which case $r(\ell_U) = \det(r)^{-1} \cdot \ell_U$) or r is a pseudoreflection through a plane perpendicular to U (in which case $r(\ell_U) = \ell_U$). Now consider the case where U is not fixed by r , letting $U_i = r^i(U)$. Since U is not fixed by r , ℓ_U has non-zero components in both W and W^\perp , so $\{U_0, U_1, \dots, U_{\text{ord}(r)-1}\}$ are distinct hyperplanes. Since $r(\ell_{U_i})$ is a linear function vanishing on U_{i+1} , we have that $r(\ell_{U_i}) = a_i \cdot \ell_{U_{i+1}}$ for some constant a_i . We have that r acts on $\prod_{i=0}^{\text{ord}(r)-1} \ell_{U_i}$ by $\prod_{i=0}^{\text{ord}(r)-1} a_i$. On the other hand, $\ell_{U_{\text{ord}(r)}} = \ell_{U_0} = r^{\text{ord}(r)}(\ell_{U_0}) = (\prod_{i=0}^{\text{ord}(r)-1} a_i) \cdot \ell_{U_{\text{ord}(r)}}$, so $\prod_{i=0}^{\text{ord}(r)-1} a_i = 1$. Summing up, if r is a pseudoreflection through a hyperplane in $H \cdot W$, then $\chi_{H \cdot W}(r) = \det(r)^{-1}$; otherwise $\chi_{H \cdot W}(r) = 1$.

Let W_1, \dots, W_k be H -orbit representatives (so every pseudoreflection hyperplane is in the H -orbit of exactly one of the W_i), and r_1, \dots, r_k be primitive pseudoreflections through the W_i , of orders e_1, \dots, e_k , respectively. Consider the map $\phi = \chi_{H \cdot W_1} \times \dots \times \chi_{H \cdot W_k} : H \rightarrow \mu_{e_1} \times \dots \times \mu_{e_k}$. By the description of $\chi_{H \cdot W_i}$ from the previous paragraph, ϕ sends r_i to a generator of μ_{e_i} , so it is surjective. Since H' is in the kernel of each $\chi_{H \cdot W_i}$, it is in the kernel of ϕ . By construction, every pseudoreflection in H is conjugate to a power of r_i for some i . Since H is generated by pseudoreflections and H/H' is abelian, every element of H/H' is represented by an element of the form $r_1^{d_1} \dots r_k^{d_k}$. Such an element is in the kernel of ϕ if and only if it is trivial, so ϕ induces an isomorphism $H/H' \rightarrow \mu_{e_1} \times \dots \times \mu_{e_k}$.

Let $V/H = \text{Spec}(k[g_1, \dots, g_n])$. Then we claim that $V/H' = \text{Spec}(k[g_1, \dots, g_n, f_{H \cdot W_1}, \dots, f_{H \cdot W_k}])$. Note first that each $f_{H \cdot W_i}$ is H' -invariant, since H acts on it by the character $\chi_{H \cdot W_i}$, and H' is in the kernel of any character. To see that this is the full ring of H' -invariants, note that any H' -invariant function is a sum of H/H' -semi-invariant functions, so it suffices to show that any H/H' -semi-invariant function f is in this ring. If each r_i acts trivially on f , then f is H -invariant, so it is some polynomial combination of the g_j . If r_i acts non-trivially on f , then f must vanish along W_i . Since f is H' -invariant, all the conjugates of r_i must also act non-trivially on f , so f must vanish along all of $H \cdot W_i$, so it must be divisible by $f_{H \cdot W_i}$, and $f/f_{H \cdot W_i}$ is H' -invariant. By induction on the degree of f , it is a polynomial in the g_j and $f_{H \cdot W_i}$.

We therefore have an H/H' -equivariant closed immersion of V/H' into \mathbb{A}^{n+k} , where μ_{e_i} acts only on the $(n + i)$ th coordinate. The following diagram is Cartesian:

$$\begin{CD} [(V/H') / (H/H')] @>>> [\mathbb{A}^{n+k} / (H/H')] \\ @VVV @VVV \\ V/H @>>> \mathbb{A}^{n+k} / (H/H'). \end{CD}$$

Since each μ_{e_i} acts on only one coordinate of \mathbb{A}^{n+k} , the right vertical arrow is a root stack morphism ramified to order e_i along the $(n + i)$ th coordinate hyperplane. It follows that the left arrow is a root stack morphism ramified to order e_i along the divisor which is the image of $H \cdot W_i$.

5 Examples, counterexamples, and questions

We noted in the introduction that as a consequence of Theorem 1, every smooth separated Deligne–Mumford stack is determined by its coarse space and the orders of its codimension 1 stabilizers. We begin this section by showing that this is not true if we drop the smoothness or separateness hypothesis, even if the exact group structure of the stabilizers is specified.

Example 1 (Different stacks with the same coarse space and stabilizers). We work over a field k with $\text{char}(k) \neq 2$.

- (1) (*Smooth separated stack*) Consider the action of $\mathbb{Z}/2$ on \mathbb{A}^1 given by $x \mapsto -x$. The quotient $\mathcal{X} = [\mathbb{A}^1 / (\mathbb{Z}/2)]$ is a smooth separated Deligne–Mumford stack with trivial generic stabilizer. Its coarse space is $X = \mathbb{A}^1$ (with coordinate x^2) and it has a single $\mathbb{Z}/2$ stabilizer at the origin.

- (2) (*Singular separated stack*) Consider the singular Deligne–Mumford stack \mathcal{X} given by the quotient of the axes in \mathbb{A}^2 by the $\mathbb{Z}/2$ -action $(x, y) \mapsto (y, x)$. The coarse space of \mathcal{X} is also \mathbb{A}^1 (with coordinate $x + y$) and the stabilizer at the origin is $\mathbb{Z}/2$.
- (3) (*Smooth non-separated stack*) Let G be the non-separated affine line. We can think of G as a group scheme over \mathbb{A}^1 via the natural map $G \rightarrow \mathbb{A}^1$. The stack BG has coarse space \mathbb{A}^1 and a stabilizer of $\mathbb{Z}/2$ at the origin. BG is a smooth Deligne–Mumford stack with trivial generic stabilizer, but is not separated (its diagonal isn't separated, so not proper). Note also that the map $BG \rightarrow \mathbb{A}^1$ is a coarse space morphism (i.e., it is a universal map from BG to an algebraic space, and is bijective on geometric points), but not a good moduli space morphism since it is not cohomologically affine. \square

In the next example, we illustrate Theorem 1 with $\mathcal{X} = [\mathbb{A}^n/S_n]$.

Example 2. Consider the stack $\mathcal{X} = [\mathbb{A}^n/S_n]$, where the S_n -action permutes the coordinates, over a field whose characteristic is prime to $n!$. The coarse space is $X = \mathbb{A}^n$, with coordinates given by the elementary symmetric functions. The S_n -action is generated by pseudoreflections corresponding to transpositions; since all transpositions are in the same conjugacy class as $(1\ 2)$, the ramification locus of the coarse space map $\mathcal{X} \rightarrow X$ is given by $(x_1 - x_2) \cap k[x_1, \dots, x_n]^{S_n}$, which is generated by the discriminant $\Delta := \prod_{i < j} (x_i - x_j)^2$, and the order of ramification is 2 (see Remark 6). The ramification divisor $D \subseteq X$ is given by expressing Δ in terms of the elementary symmetric functions. Up to sign, this is given by the resultant of f and f' , where $f(x) = x^n - e_1x^{n-1} + \dots + (-1)^ne_n$ is the polynomial with the elementary symmetric functions as coefficients.

Since the action of S_n on \mathbb{A}^n is the sum of the trivial and standard representations of S_n , the features of this example are all present if one restricts to the vanishing locus of $e_1 = x_1 + \dots + x_n$, the first elementary symmetric function. In this case, the coarse space of $[\mathbb{A}^{n-1}/S_n]$ is $\mathbb{A}^{n-1} = \text{Spec } k[e_2, \dots, e_n]$, the ramification divisor is given by the discriminant of $f(x) = x^n + e_2x^{n-2} + \dots + (-1)^ne_n$, and the ramification order is 2. \square

Example 3 (Square root of the cuspidal cubic). In this example, we construct the smooth stack \mathcal{X} with coarse space $\mathbb{A}^2 = \text{Spec } \mathbb{C}[a, b]$ so that the ramification divisor is the cuspidal cubic $D = V(a^3 - b^2)$ with ramification order 2. The coarse space is already smooth, so it is equal to its canonical stack. The square root of the cuspidal cubic is $\sqrt{D}/\mathbb{A}^2 = [Y/(\mathbb{Z}/2)]$, where $Y = \text{Spec } \mathbb{C}[a, b, t]/(b^3 - a^2 - t^2)$ is the A_2 -singularity, and $\mathbb{Z}/2$

acts as $t \mapsto -t$. The canonical stack of Y is $Y^{\text{can}} = [(\text{Spec } \mathbb{C}[x, y]) / (\mathbb{Z}/3)]$ where the $\mathbb{Z}/3$ -action is given by $(x, y) \mapsto (\zeta x, \zeta^2 y)$ with ζ a third root of unity. The relation between the x, y coordinates and the a, b, t coordinates is $x^3 = a + ti, y^3 = a - ti$, and $xy = b$. We see that the $\mathbb{Z}/2$ -action on Y fixes xy and swaps x^3 and y^3 . It therefore lifts to an action on \mathbb{A}^2 swapping x and y . Together with the $\mathbb{Z}/3$ -action, this lifted $\mathbb{Z}/2$ -action generates an action of S_3 on \mathbb{A}^2 . This S_3 -action is the unique two-dimensional irreducible representation. We therefore see that $\mathcal{X} = [Y / (\mathbb{Z}/2)]^{\text{can}} = [Y^{\text{can}} / (\mathbb{Z}/2)] = [\mathbb{A}^2 / S_3]$. This is the $n = 3$ case of Example 2. □

Remark 4. The above example can be done over \mathbb{R} instead of over \mathbb{C} , but since \mathbb{R} has no primitive cube root of unity, the stabilizer of the origin is $(\mathbb{Z}/2) \times \mu_3$, a twisted version of S_3 . To get a stabilizer of S_3 , one would square root the cuspidal cubic given by the discriminant $\Delta = 4a^3 - 27b^2$. □

Example 5 (Reducible ramification divisor without simple normal crossings). Let k be a field with primitive fourth roots of unity and with $\text{char}(k) \neq 2$. Consider $D \subseteq \mathbb{A}^2$ defined by $y(x^2 - y) = 0$. Then $\sqrt{D/\mathbb{A}^2} = [Y / (\mathbb{Z}/2)]$ where $Y = \text{Spec } k[x, y, t] / (y^2 - x^2y + t^2)$ and $\mathbb{Z}/2$ acts as $t \mapsto -t$. Letting $z = y - \frac{1}{2}x^2$, we see that the defining equation of Y is $\frac{1}{4}x^4 = z^2 + t^2$. After a further change of coordinates $u = 2(z + ti)$ and $v = 2(z - ti)$, we see that Y is the A_3 -singularity defined by the equation $x^4 = uv$. In these coordinates, the $\mathbb{Z}/2$ -action on Y fixes x and swaps u and v . Since Y is the A_3 -singularity, $Y^{\text{can}} = [\mathbb{A}^2 / (\mathbb{Z}/4)]$ where $\mathbb{Z}/4$ acts on \mathbb{A}^2 by $(a, b) \mapsto (ia, -ib)$. The relation between the a, b coordinates and the x, u, v coordinates is given by $x = ab, u = a^4$, and $v = b^4$. The $\mathbb{Z}/2$ -action on Y lifts to \mathbb{A}^2 by swapping a and b . With the action of $\mathbb{Z}/4$, this generates an action of the dihedral group D_8 , and so $\sqrt{D/\mathbb{A}^2}^{\text{can}} = [\mathbb{A}^2 / D_8]$. This argument generalizes to show that $\sqrt{V(Y(x^n - y)) / \mathbb{A}^2}^{\text{can}} = [\mathbb{A}^2 / D_{4n}]$. □

Example 6 (Singular coarse space). Let $X = \text{Spec } k[a, b, c] / (ab - c^m)$ be the A_{m-1} -singularity and let $D \subseteq X$ be a Weil divisor which generates the class group of X . The canonical stack is $X^{\text{can}} = [\text{Spec}(k[x, y]) / \mu_m]$ where the action is $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1}y)$, with $(a, b, c) = (x^m, y^m, xy)$. D lifts to the divisor $[V(x) / \mu_m]$. The root stack $\sqrt[n]{D/X^{\text{can}}}$ is then

$$\begin{aligned} \sqrt[n]{[V(x) / \mu_m] / [\text{Spec } k[x, y] / \mu_m]} &= [\sqrt[n]{V(x) / \text{Spec } k[x, y]} / \mu_m] \\ &= [[\text{Spec}(k[x, y, t] / (x - t^n)) / \mu_n] / \mu_m], \end{aligned}$$

where μ_n acts as $\zeta \cdot (x, y, t) = (x, y, \zeta t)$. This stack is the quotient $[\mathrm{Spec}(k[y, t])/\mu_n]$, where the action is $\zeta \cdot (y, t) = (\zeta^{-1}y, \zeta^m t)$. \square

Given a smooth separated Deligne–Mumford stack \mathcal{X} with trivial generic stabilizer, we can associate to it the tuple (X, D_i, e_i) , where X is its coarse space, the D_i are the irreducible components of the ramification divisor of the coarse space map $\pi : \mathcal{X} \rightarrow X$, and e_i is ramification degree of π over D_i . Although our main result shows that \mathcal{X} is characterized by (X, D_i, e_i) , we now show that not every tuple (X, D_i, e_i) arises in this manner.

Example 7 (Not every tuple (X, D_i, e_i) arises). Consider the cone $D = V(xy + z^2)$ in $X = \mathbb{A}^3$. Taking the square root of X along D yields a stack $\tilde{\mathcal{X}}$ with a singularity of the form $xy + z^2 = t^2$. This is isomorphic to the toric singularity $xy = uv$ (with $(u, v) = (t + z, t - z)$), which is not a finite quotient singularity (and hence has no canonical Deligne–Mumford stack). It follows from Theorem 1 that there is no smooth separated Deligne–Mumford stack \mathcal{X} with trivial generic stabilizer and coarse space X whose coarse space morphism $\mathcal{X} \rightarrow X$ is ramified over D with degree 2. \square

In light of this example, we ask:

Question 1. Which tuples (X, D_i, e_i) arise in the manner described above? That is, given an algebraic space X with tame quotient singularities, a reduced Weil divisor D on X with irreducible components D_i , and integers $e_i \geq 2$, is there a smooth separated Deligne–Mumford stack \mathcal{X} with a coarse space morphism $\mathcal{X} \rightarrow X$ ramified over D_i with degree e_i ? \square

Remark 8. It suffices to answer Question 1 formally locally: which singularity types and ramification orders can arise for the quotient of a vector space V by a faithful linear action a finite group G ? By Remark 6, the ramification divisors are the images of the hyperplanes of pseudoreflections, and the ramification degrees are the orders of the pseudoreflections. \square

Example 9 (Even if (X, D_i, e_i) arises, (X, D_i, ne_i) may not). Let $D \subseteq \mathbb{A}_\mathbb{C}^2$ be the cuspidal cubic $x^3 - y^2 = 0$. The root stack $\sqrt[n]{D/X}$ has a unique singularity, which is of the form $x^3 - y^2 = z^n$. For $n = 2, 3, 4, 5$, this is a singularity of type A_2, D_4, E_6 , and E_8 , respectively, (see e.g., [13, Section 4.2] for descriptions of these singularities). (It is immediate that $x^3 - y^2 = z^n$ is the A_2, E_6 , and E_8 singularity for $n = 2, 4, 5$, respectively. To see that $x^3 - y^2 = z^3$

is the D_4 -singularity, make the coordinate change $(u, v) = (\sqrt[3]{4/7}(x+z), \sqrt[6]{4/189} \frac{1}{2}(z-x))$. In general, for $n \geq 6$, the singularity $x^3 - y^2 = z^n$ is not a quotient singularity; for $6 \leq n \leq 11$, this singularity is elliptic [13, Exercise 18, Chapter 4]. Thus, (\mathbb{A}^2, D, n) arises as the coarse space and ramification divisor of a smooth separated Deligne–Mumford stack for $n \leq 5$, but in general not for $n \geq 6$. \square

Given the above example, we pose the following question.

Question 2. If X has tame quotient singularities and $D \subseteq X$ is an irreducible divisor which does not have simple normal crossings, does (X, D, n) fail to arise for all sufficiently large n ? \square

Finally, we mention that the tameness assumption in Theorem 1 is needed. The following example is due to David Zureick-Brown.

Example 10 (Tameness assumption is necessary). Let k be an algebraically closed field of characteristic p and consider the Artin–Schreier curve $\phi : C \rightarrow \mathbb{P}^1$ corresponding to the field extension $k(C) = k(x)[y]/(y^p - y - f(x))$ of $k(x)$. Since $k(C)$ is a Galois extension of $k(x)$ with Galois group \mathbb{Z}/p , we know that ϕ is a generic \mathbb{Z}/p -torsor. Specifically, the action of $\lambda \in \mathbb{Z}/p$ is given by $(x, y) \mapsto (x, y + \lambda)$ and one checks that ϕ is totally ramified over ∞ and is a \mathbb{Z}/p -torsor over \mathbb{A}^1 . Thus, taking the stack quotient, we obtain a smooth separated non-tame Deligne–Mumford stack $\mathcal{X} = [C/(\mathbb{Z}/p)]$ and coarse space map $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$ which is an isomorphism over \mathbb{A}^1 . Regardless of f , the stack \mathcal{X} has a single \mathbb{Z}/p -stabilizer over ∞ ; however different choices of f can yield non-isomorphic \mathcal{X} , and in fact C of different genus. So, we see that if we drop the tameness assumption in the statement of Theorem 1, then the ramification divisor and degrees of ramification are not enough to recover \mathcal{X} . \square

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