HEIGHT GAP CONJECTURES, D-FINITENESS, AND A WEAK DYNAMICAL MORDELL–LANG CONJECTURE

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Abstract. In previous work, the first author, Ghioca, and the third author introduced a broad dynamical framework giving rise to many classical sequences from number theory and algebraic combinatorics. Specifically, these are sequences of the form $f(\Phi^n(x))$, where $\Phi\colon X \dashrightarrow X$ and $f\colon X \dashrightarrow \mathbb{P}^1$ are rational maps defined over $\overline{\mathbb{Q}}$ and $x \in X(\overline{\mathbb{Q}})$ is a point whose forward orbit avoids the indeterminacy loci of Φ and f. They conjectured that if the sequence is infinite, then $\limsup \frac{h(f(\Phi^n(x)))}{\log n} > 0$. They also made a corresponding conjecture for $\liminf$ and showed that it implies the Dynamical Mordell–Lang Conjecture. In this paper, we prove the $\limsup$ conjecture as well as the $\liminf$ conjecture away from a set of density 0. As applications, we prove results concerning the height growth rate of coefficients of D-finite power series as well as the Dynamical Mordell–Lang Conjecture up to a set of density 0.

1. Introduction

In [BGS], the authors introduced a broad dynamical framework giving rise to many classical sequences from number theory and algebraic combinatorics. In particular, this construction yields all sequences whose generating functions are D-finite, i.e., those satisfying homogeneous linear differential equations with rational function coefficients. This class, in turn, contains all hypergeometric series (see, e.g., [WZ92, Gar09]), all series related to integral factorial ratios [Bob09, Sou19], generating functions for many classes of lattice walks [DHRS18], diagonals of rational functions [Lip88], algebraic functions [Lip89], generating series for the cogrowth of many finitely presented groups [GP17], as well as generating functions of numerous classical combinatorial sequences (see Stanley [Sta99, Chapter 6] and the examples therein). In [BGS], they also stated the so-called $\limsup$ and $\liminf$ Height Gap Conjectures, which if true, would imply both the Dynamical Mordell–Lang Conjecture as well as results concerning the height growth rate of coefficients of D-finite power series. The goal of this paper is to prove a uniform version of the $\limsup$ Height Gap Conjecture and to prove the $\liminf$ version away from a set of density zero. Consequently, we obtain applications to D-finite power series and a weak version of the Dynamical Mordell–Lang Conjecture.

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To state our results, we fix the following notations. Throughout, we let $\mathbb{N}$ (resp. $\mathbb{Z}^+$) denote the set of all nonnegative (resp. positive) integers. Let h denote the absolute logarithmic Weil height function. We refer the reader to [Lan83, HS00, BG06] for the theory of Weil heights. Given an arbitrary rational map g, let I_g denote its indeterminacy locus. If Φ is a rational self-map of a quasi-projective variety X defined over $\overline{\mathbb{Q}}$, then we let $X_\Phi(\overline{\mathbb{Q}})$ denote the subset of points $x \in X(\overline{\mathbb{Q}})$ such that for any $n \in \mathbb{N}$, the n-th iterate $\Phi^n(x)$ avoids I_Φ; for such an $x \in X_\Phi(\overline{\mathbb{Q}})$, we let $\mathcal{O}_\Phi(x)$ denote its forward orbit under Φ. Lastly, if $f\colon X \dashrightarrow \mathbb{P}^1$ is a rational function, let $X_{\Phi,f}(\overline{\mathbb{Q}}) \subseteq X_\Phi(\overline{\mathbb{Q}})$ be the subset of points x with $I_f \cap \mathcal{O}_\Phi(x) = \varnothing$.

The following lim sup Height Gap Conjecture was introduced in [BGS].

Conjecture 1.1 (cf. [BGS, Conjecture 1.4]). *Let X be a quasi-projective variety, let $\Phi\colon X \dashrightarrow X$ be a rational self-map, and let $f\colon X \dashrightarrow \mathbb{P}^1$ be a non-constant rational function, all defined over $\overline{\mathbb{Q}}$. Then for any $x \in X_{\Phi,f}(\overline{\mathbb{Q}})$, either $f(\mathcal{O}_\Phi(x))$ is finite, or*

$$\limsup_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log n} > 0.$$

Our first main result is a simple proof of Conjecture 1.1. This generalizes [BGS, Theorem 1.3], which handled the case where Φ and f are morphisms.

Theorem 1.2 (lim sup Height Gap). *Conjecture 1.1 is true.*

In [BGS], the authors also introduced the following lim inf Height Gap Conjecture and showed that it implies the Dynamical Mordell–Lang Conjecture ([BGT16]).

Conjecture 1.3 (cf. [BGS, Conjecture 1.6]). *Let X, Φ, f, and x be as in Conjecture 1.1. Suppose further that X is irreducible and the orbit $\mathcal{O}_\Phi(x)$ is Zariski dense in X. Then*

$$\liminf_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log n} > 0.$$

Generalizing our method of proof of Theorem 1.2 via a more involved technique introduced in Section 3, we obtain a uniform version of the above lim sup height gap result for any subset $T \subseteq \mathbb{N}$ of positive density (see Definition 3.1 for the notion of upper asymptotic density).

Theorem 1.4 (Uniform lim sup Height Gap). *Let X, Φ, f, and x be as in Conjecture 1.1. Then either f is constant on some periodic component of the Zariski closure of $\mathcal{O}_\Phi(x)$, or there exists an $\epsilon > 0$ such that for any subset $T \subseteq \mathbb{N}$ of positive density, we have*

$$\limsup_{n\in T} \frac{h(f(\Phi^n(x)))}{\log n} > \epsilon.$$

The significance of our above uniform bound is that it immediately implies the lim inf Height Gap Conjecture 1.3 away from a set of density zero.

Theorem 1.5 (Weak lim inf Height Gap). *Let X, Φ, f, and x be as in Conjecture 1.1. If f is non-constant on each periodic component of the Zariski closure of $\mathcal{O}_\Phi(x)$, then there exists a constant $C > 0$ and a set $S \subset \mathbb{N}$ of upper asymptotic density zero such that*

$$h(f(\Phi^n(x))) > C \log n$$

whenever $n \notin S$, or equivalently,

$$\liminf_{n \in \mathbb{N} \setminus S} \frac{h(f(\Phi^n(x)))}{\log n} > 0.$$

Note that Conjecture 1.3 is stated only when X is irreducible and $\mathcal{O}_\Phi(x)$ is Zariski dense in X because, otherwise, an example of [BGS] with non-Zariski dense orbit shows that it is false. However, our result holds under a weaker but necessary hypothesis that f is non-constant on each periodic component of the Zariski closure of $\mathcal{O}_\Phi(x)$.

Remark 1.6. We observe that variants of our main results (i.e., Theorems 1.2, 1.4, and 1.5) also hold for function fields over finite fields. Our proofs are characteristic-free except for our use of Schanuel's Theorem 2.1 for number fields (e.g., in Step 3 of our proof of Theorem 1.4). To obtain the result for function fields, one requires analogues of Schanuel's Theorem. In the global case (i.e., function fields of curves over finite fields), an analogue of Schanuel's Theorem due to Wan [Wan92] can be used to obtain gaps of the same type as in the number field case. For function fields of higher transcendence degree over finite fields, however, it appears that exact asymptotics have not been explicitly worked out in the literature. Nevertheless, we give estimates in Remark 3.11, which show the types of gaps one can attain via our methods.

For function fields of arbitrary transcendence degree over infinite base fields, it turns out to be a difficult problem to obtain the types of gaps that we obtain in the case of number fields. For example, Cantat and Xie [CX20] show that for birational maps Φ of projective spaces, either the sequence $(\max_{0 \le i \le n} \deg \Phi^i)_{n \in \mathbb{N}}$ is bounded, or it grows asymptotically faster than the inverse of the diagonal Ackermann function which has significantly slower growth than $\log n$. Although our methods show that for function fields over finite fields one gets the same type of gap result as in the number field case (see Remark 3.11), examples suggest that a stronger gap result might in fact hold in the general function field case. Inspired by the recent results of Cantat and Xie and others (see [CX20] and references therein), we pose the following height gap question.

Question 1.7. *Let K be a function field of transcendence degree $d \ge 1$ over a field $\Bbbk$ of arbitrary characteristic. Let X be an irreducible quasi-projective variety, let $\Phi \colon X \dashrightarrow X$ be a rational self-map, and let $f \colon X \dashrightarrow \mathbb{P}^1$ be a non-constant rational function, all defined over K. Suppose that $x \in X_{\Phi,f}(K)$ so that $h(f(\Phi^n(x)))$ is not uniformly bounded. Then is there a constant $\epsilon > 0$ such that*

$$\limsup_{n \to \infty} \frac{h(f(\Phi^n(x)))}{n} > \epsilon\,?$$

Remark 1.8. One may also like to consider a lim inf version of this question. However, the naive dichotomy does not hold for function fields in positive characteristic. For instance, let X be the affine plane $\mathbb{A}^2$ over $\overline{\mathbb{F}_p(t)}$ with coordinates x_1, x_2, let $\Phi\colon X \longrightarrow X$ be a self-map of X defined by $(x_1, x_2) \mapsto (tx_1, (1+t)x_2)$, and let $f\colon X \longrightarrow \mathbb{A}^1$ be the function $f(x_1, x_2) = x_2 - x_1$. Then $f(\Phi^n(1,1)) = 1$ whenever n is a power of p, but if $\gcd(n,p) = 1$ then $f(\Phi^n(1,1))$ has degree $n-1$. It follows that the sequence $h(f(\Phi^n(1,1)))$ is unbounded but $\liminf_{n\to\infty} h(f(\Phi^n(1,1))) = 0$. This example is essentially due to Lech [Lec53].

We provide two applications of our main results. As an application of Theorem 1.2, we obtain a simple proof of the univariate version of a result of Bell–Nguyen–Zannier [BNZ] which, in turn, generalized results of van der Poorten–Shparlinski [vdPS96] with the aid of Bell–Chen [BC17].

Recall that a power series $F(z) \in \overline{\mathbb{Q}}[[z]]$ is *D-finite*, if it is the solution of a non-trivial homogeneous linear differential equation with coefficients in the rational function field $\overline{\mathbb{Q}}(z)$; this is equivalent to saying that the coefficients of $F(z)$ satisfy certain linear recurrence relations with polynomial coefficients (see [Sta80, Theorem 1.5]).

Theorem 1.9 (Height gaps for D-finite power series)**.** *If $\sum_{n\geq 0} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is D-finite and $\limsup\limits_{n\to\infty} \dfrac{h(a_n)}{\log n} = 0$, then the sequence $(a_n)_{n\in\mathbb{N}}$ is eventually periodic.*

As an application of Theorem 1.5, we show that the Dynamical Mordell–Lang Conjecture holds away from a set of upper asymptotic density zero (see Definition 3.1). We note that a slightly stronger version of this result has been obtained in [BGT15, Corollary 1.5] using the upper Banach density function (which is not less than our upper asymptotic density).

Theorem 1.10 (Weak Dynamical Mordell–Lang)**.** *Let X be a quasi-projective variety, $\Phi\colon X \dashrightarrow X$ a rational self-map, and $Y \subseteq X$ a subvariety of X, all defined over $\overline{\mathbb{Q}}$. If $x \in X_\Phi(\overline{\mathbb{Q}})$, then $\{n \in \mathbb{N} : \Phi^n(x) \in Y\}$ is a union of finitely many arithmetic progressions along with a set of upper asymptotic density zero.*

Lastly, we prove a generalization of Theorem 1.5 for multiple commuting rational self-maps (under slightly stronger assumptions). We fix some notations first. Given m commuting rational self-maps $\Phi_1, \ldots, \Phi_m$ of X and $\mathbf{n} \coloneqq (n_1, \ldots, n_m) \in \mathbb{N}^m$, let $\Phi^{\mathbf{n}}$ denote the composite $\Phi_1^{n_1} \circ \cdots \circ \Phi_m^{n_m}$; let $X_{\Phi_1,\ldots,\Phi_m}(\overline{\mathbb{Q}})$ denote the subset of points $x \in X(\overline{\mathbb{Q}})$ such that for every $\mathbf{n} \in \mathbb{N}^m$, the $\mathbf{n}$-th iterate $\Phi^{\mathbf{n}}(x)$ avoids the indeterminacy loci of all $\Phi_1, \ldots, \Phi_m$. For any $x \in X_{\Phi_1,\ldots,\Phi_m}(\overline{\mathbb{Q}})$, as usual, $\mathcal{O}_{\Phi_1,\ldots,\Phi_m}(x)$ stands for the orbit of x under all $\Phi_1, \ldots, \Phi_m$, i.e., the set of points of the form $\Phi^{\mathbf{n}}(x)$. Similarly, for a rational function $f\colon X \dashrightarrow \mathbb{P}^1$, let $X_{\Phi_1,\ldots,\Phi_m,f}(\overline{\mathbb{Q}}) \subseteq X_{\Phi_1,\ldots,\Phi_m}(\overline{\mathbb{Q}})$ denote the subset of points x with $I_f \cap \mathcal{O}_{\Phi_1,\ldots,\Phi_m}(x) = \varnothing$. We endow $\mathbb{N}^m$ with the 1-norm $\|\mathbf{n}\|_1 \coloneqq n_1 + \cdots + n_m$.

Theorem 1.11 (Weak lim inf Height Gap for multiple maps)**.** *Let X be an irreducible quasi-projective variety, let $\Phi_1, \ldots, \Phi_m$ be m commuting rational self-maps of X, and let $f\colon X \dashrightarrow \mathbb{P}^1$ be a non-constant rational function, all defined over $\overline{\mathbb{Q}}$. If the orbit*

$\mathcal{O}_{\Phi_1,\ldots,\Phi_m}(x)$ of $x \in X_{\Phi_1,\ldots,\Phi_m,f}(\overline{\mathbb{Q}})$ is Zariski dense in X, then there exist a constant $C > 0$ and a set $S \subset \mathbb{N}$ of upper asymptotic density zero such that

$$\max_{\{\mathbf{n}\,:\,\|\mathbf{n}\|_1 = n\}} h(f(\Phi^{\mathbf{n}}(x))) > C \log n$$

whenever $n \notin S$, or equivalently,

$$\liminf_{n \in \mathbb{N} \setminus S} \max_{\{\mathbf{n}\,:\,\|\mathbf{n}\|_1 = n\}} \frac{h(f(\Phi^{\mathbf{n}}(x)))}{\log n} > 0.$$

Theorem 1.11 shall be proved by induction on the number of self-maps for all X, where the base case $m = 1$ is implied by Theorem 1.5. In Example 5.1, we show that one cannot expect a version of Theorem 1.11 to hold if one takes the $\liminf$ over $\mathbb{N}^m$ except a subset $\mathbf{S}$ of density zero, nor if one does not take the maximum over $\mathbf{n} \in \mathbb{N}^m$ with $\|\mathbf{n}\|_1 = n$.

In [Lip89], Lipshitz introduced and studied multivariate D-finite power series, which extended Stanley's pioneering work [Sta80] on univariate D-finite power series. Recently, the first author, Nguyen, and Zannier proved a height gap result for the coefficients of multivariate D-finite power series; see [BNZ, Theorem 1.3(c)]. The reader may be curious to know whether it is possible to deduce their result from Theorem 1.11, analogously to how we deduced the univariate D-finiteness result Theorem 1.9 from Theorem 1.2. This appears to be a subtle issue: our proof of Theorem 1.9 relies on the fact that for sufficiently large n, the coefficients of a univariate D-finite power series are of the form $f(\Phi^n(c))$ for certain choices of X, Φ, f, and c; see [BGT16, §3.2.1]. In contrast, we construct in Example 5.2 a D-finite power series in two variables (in fact, a rational function) whose coefficients never arise as $f(\Phi_1^{n_1} \circ \Phi_2^{n_2}(c))$ for any choices of X, Φ_1, Φ_2, f, and c.

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2. The $\limsup$ Height Gap: Proof of Theorem 1.2

We start by recalling Schanuel's Theorem, which plays a central role in the proofs of Theorems 1.2 and 1.4. It can be regarded as a quantitative version of Northcott's theorem. Schanuel's Theorem has a conjectural extension to Fano varieties, known as Manin's conjecture, which has attracted a lot of attention recently (see the survey [LT19] and references therein).

Theorem 2.1 (Schanuel [Sch79], cf. [HS00, Theorem B.6.2] or [BG06, 11.10.5])**.** *Let K be a number field of degree d and let h denote the absolute logarithmic Weil height function. Then we have*

$$\lim_{B \to \infty} \frac{\#\{P \in \mathbb{P}^n(K) : h(P) \le \log B\}}{B^{d(n+1)}} = C(n, K) > 0,$$

where the positive constant $C(n, K)$ depends only on n and K.

We shall prove Theorem 1.2 via an application of Schanuel's Theorem 2.1 and the following lemma. Recall that a topological space U is called *Noetherian* if the descending chain condition holds for closed subsets of U, i.e., for every chain of closed sets $Z_1 \supset Z_2 \supset \dots$, there is some $m \geq 1$ for which $Z_m = Z_n$ for all $n \geq m$.

Lemma 2.2. *Let X be a quasi-projective variety, let $\Phi\colon X \dashrightarrow X$ be a rational self-map, and let $f\colon X \dashrightarrow \mathbb{P}^1$ be a rational function, all defined over $\overline{\mathbb{Q}}$. Then there exists a constant $\ell \in \mathbb{N}$ with the following property: if $x, y \in X_{\Phi,f}(\overline{\mathbb{Q}})$ and $f(\Phi^n(x)) = f(\Phi^n(y))$ for $0 \leq n \leq \ell$, then $f(\Phi^n(x)) = f(\Phi^n(y))$ for all $n \geq 0$.*

Proof. Let $U_n = X \smallsetminus \bigcup_{j \leq n}(I_{\Phi^j} \cup I_{f\circ\Phi^j})$ and $U = \bigcap_n U_n$. By construction, the $\overline{\mathbb{Q}}$-points of U are precisely those on which Φ^n and $f \circ \Phi^n$ are well-defined for all $n \geq 0$, i.e., $U(\overline{\mathbb{Q}}) = X_{\Phi,f}(\overline{\mathbb{Q}})$. We endow U with the subspace topology inherited from X thereby making it a Noetherian topological space. Since

$$U \times U \hookrightarrow U_n \times U_n \xrightarrow{(f\circ\Phi^n,\, f\circ\Phi^n)} \mathbb{P}^1 \times \mathbb{P}^1$$

is continuous and the image of the diagonal map $\mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is closed, we see that

$$Z_n := \{(x, y) \in U \times U : f(\Phi^i(x)) = f(\Phi^i(y)) \text{ for } i \leq n\}$$

is a closed subset of $U \times U$. As $U \times U$ is Noetherian, there exists an $\ell \in \mathbb{N}$ such that $Z_n = Z_\ell$ for all $n \geq \ell$. $\square$

Proof of Theorem 1.2. Let $x \in X_{\Phi,f}(\overline{\mathbb{Q}})$. Without loss of generality, we may assume that X, Φ, f, and x are defined over a fixed number field K. Suppose that

$$\limsup_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log n} = 0,$$

i.e., $h(f(\Phi^n(x))) = o(\log n)$. We will show that $f(\mathcal{O}_\Phi(x))$ is finite.

Letting ℓ be as in Lemma 2.2, we define

$$y_i := (f(\Phi^i(x)), f(\Phi^{i+1}(x)), \dots, f(\Phi^{i+\ell}(x))) \in (\mathbb{P}^1)^{\ell+1}(K)$$

for $i \geq 0$, and let $S = \{y_i : i \geq 0\}$. Via the Segre embedding, we may view $S \subseteq \mathbb{P}^{2^{\ell+1}-1}(K)$. Then

$$h(y_i) = \sum_{j=0}^{\ell} h(f(\Phi^{i+j}(x))) = o(\log i).$$

Next, choose $0 < \epsilon < ([K : \mathbb{Q}]2^{\ell+1})^{-1}$. Then there exists $N_0 \in \mathbb{Z}^+$ such that for all $i \geq N_0$, we have $h(y_i) < \epsilon \log i$. So, for all $n \geq N_0$,

$$\#\{y_{N_0}, y_{N_0+1}, \dots, y_n\} \leq \#\{z \in \mathbb{P}^{2^{\ell+1}-1}(K) : h(z) \leq \log n^\epsilon\} = O(n^{\epsilon[K:\mathbb{Q}]2^{\ell+1}}),$$

where the equality comes from applying Schanuel's Theorem 2.1. Choosing n sufficiently large, we find

$$\#\{y_{N_0}, y_{N_0+1}, \dots, y_n\} < n - N_0.$$

In particular, there exist $i < j$ for which $y_i = y_j$. Thus, $f(\Phi^n(\Phi^i(x))) = f(\Phi^n(\Phi^j(x)))$ for all $0 \leq n \leq \ell$, and so Lemma 2.2 implies $f(\Phi^{n+i}(x)) = f(\Phi^{n+j}(x))$ for all $n \geq 0$. It follows that $f(\Phi^n(x))$ is eventually periodic with period dividing $j-i$. Hence, $f(\mathcal{O}_\Phi(x))$ is finite. $\square$

3. Uniform lim sup Height Gap: Proof of Theorem 1.4

The main goal of this section is to prove Theorem 1.4 which is the strengthening of Theorem 1.2.

3.1. Preliminary results on sets of positive density.

Definition 3.1. Let A be a subset of $\mathbb{Z}^+$. The *upper asymptotic (or natural) density* $\overline{d}(A)$ of A is defined by

$$\overline{d}(A) := \limsup_{m\to\infty} \frac{|A\cap[1,m]|}{m}.$$

We frequently refer to $\overline{d}(A)$ simply as the *density* of A.

Remark 3.2. It is easy to see that the density $\overline{d}(A)$ of any $A \subseteq \mathbb{Z}^+$ is *right translation invariant*, i.e., $\overline{d}(A+i) = \overline{d}(A)$ for any $i \in \mathbb{N}$, where $A+i := \{a+i : a \in A\}$. Consequently, can extend the definition of density to any $A \subseteq \mathbb{Z}$ that is bounded from below.

Remark 3.3. Let $T \subseteq \mathbb{N}$ have positive density and let $L \geq 1$. By the subadditivity of natural density, there exists some $a \in \{0, 1, \dots, L-1\}$ such that $T \cap (a + L\mathbb{N})$ has positive density.

Definition 3.4. Given $T \subseteq \mathbb{N}$, the *shift set* of T is defined to be

$$\Sigma(T) = \{i \in \mathbb{N} : \overline{d}(T \cap (T+i)) > 0\}.$$

Our goal in this subsection is to prove that if T has positive density, then $\Sigma(T)$ does as well. We prove this after a preliminary lemma.

Lemma 3.5. *Let $T \subseteq \mathbb{N}$ and $N \in \mathbb{Z}^+$ satisfying $\overline{d}(T) > \frac{1}{N}$. Then for any finite subset $F \subseteq \mathbb{N}$ with $|F| \geq N$, there exist $j, k \in F$ with $j > k$ such that $\overline{d}((T + (j-k)) \cap T) > 0$.*

Proof. For ease of notation, we let $T_i = T + i$ for any $i \in \mathbb{N}$. By definition of the density function, there is a sequence $0 < m_1 < m_2 < \cdots < m_n < \cdots$ and intervals $I_n = [0, m_n] \subset \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{|T \cap I_n|}{|I_n|} = \overline{d}(T).$$

For each $i \in \mathbb{N}$, we have $|T \cap I_n| - i \leq |T \cap (I_n - i)| \leq |T \cap I_n|$, and so $\lim\limits_{n\to\infty} \dfrac{|T \cap (I_n - i)|}{|I_n|} = \overline{d}(T)$. In particular, this holds for each $i \in F$.

Fix an $\epsilon > 0$ with $\frac{1}{N} + \epsilon < \overline{d}(T)$. Then for n sufficiently large,

$$\frac{|T_i \cap I_n|}{|I_n|} = \frac{|T \cap (I_n - i)|}{|I_n|} > \frac{1}{N} + \epsilon$$

for all $i \in F$. Now, suppose to the contrary that $\overline{d}(T_j \cap T_k) = \overline{d}(T_{j-k} \cap T) = 0$ for all distinct $j, k \in F$ with $j > k$. It follows that for n sufficiently large,

$$|T_j \cap T_k \cap I_n| < \frac{2|I_n|}{|F|}\epsilon$$

for all distinct $j, k \in F$. Clearly,

$$|I_n| \geq \big|I_n \cap \bigcup_{i\in F} T_i\big| = \big|\bigcup_{i\in F}(T_i \cap I_n)\big|.$$

However, the inclusion-exclusion principle asserts that

$$\begin{aligned}\big|\bigcup_{i\in F}(T_i \cap I_n)\big| &\geq \sum_{i\in F}|T_i \cap I_n| - \sum_{\substack{j,k\in F\\ j>k}}|T_j \cap T_k \cap I_n| \\ &> |F|\big(\frac{1}{N} + \epsilon\big)|I_n| - \binom{|F|}{2}\frac{2|I_n|}{|F|}\epsilon \\ &= \Big(\frac{|F|}{N} + \epsilon\Big)|I_n| \geq (1+\epsilon)|I_n|,\end{aligned}$$

which yields a contradiction and hence Lemma 3.5 follows. □

The following result is strengthening of [BGT15, Lemma 2.1].

Proposition 3.6. *If $T \subseteq \mathbb{N}$ satisfies $\overline{d}(T) > 0$, then $\overline{d}(\Sigma(T)) > 0$.*

Proof. Choose a positive integer N satisfying $\overline{d}(T) > \frac{1}{N}$, and let T_i denote $T + i$ for any $i \in \mathbb{N}$. If $\Sigma(T) = \mathbb{N}$, then there is nothing to prove. So we may suppose there is some $i \in \mathbb{N}$ such that $\overline{d}(T \cap T_i) = 0$. Consider the set $\mathcal{S}$ of those finite subsets $F \subseteq \mathbb{N}$ such that $\overline{d}(T_{j-k} \cap T) = 0$ for all $j, k \in F$ with $j > k$. Clearly, $\mathcal{S} \neq \varnothing$ as $\{1, i+1\} \in \mathcal{S}$. Moreover, by Lemma 3.5, we know $|F| < N$ for any $F \in \mathcal{S}$.

Let $\varnothing \neq F_{\max} \subseteq \mathbb{N}$ be any maximal element of $\mathcal{S}$ (with respect to inclusion of sets), and let M be the maximum element of $F_{\max}$. Then by our definition of $F_{\max}$, for any integer $n > M$, there exists some $k_n \in F_{\max}$ satisfying

$$\overline{d}(T_{n-k_n} \cap T) > 0, \text{ i.e., } n - k_n \in \Sigma(T).$$

Since $0 \leq k_n \leq M$, we see $n - M \leq n - k_n \leq n$. In particular, for every $c \geq 2$, we have

$$iM - k_{cM} \in \Sigma(T) \cap [(c-1)M, cM].$$

It thus follows from the definition of density that $\overline{d}(\Sigma(T)) \geq \lim\limits_{c\to\infty} \dfrac{c-1}{cM} = \dfrac{1}{M}$. □

Remark 3.7. Using a similar argument, one can obtain an analogue of Proposition 3.6 where one replaces $\overline{d}$ by upper Banach density.

3.2. **Stable non-periodic dimension.** Given a Noetherian topological space U of finite Krull dimension and continuous map $\Phi\colon U \longrightarrow U$, a subset $Y \subseteq U$ is *periodic with respect to* Φ if $\Phi^n(Y) \subseteq Y$ for some positive integer n; we frequently say Y is Φ-*periodic* or simply *periodic* if Φ is understood from context. If $Z \subseteq U$ is a closed subset, let $Z_1, \ldots, Z_r$ denote its irreducible components and consider the set

$$\mathcal{S} = \left\{ \bigcup_{i \in I} Z_i : I \subseteq \{1, \ldots, r\} \right\}.$$

Notice that if $Y_1, Y_2 \in \mathcal{S}$ are periodic with respect to Φ, then so is $Y_1 \cup Y_2$. In particular, there is a unique maximal Φ-periodic element of $\mathcal{S}$ which we denote by $P_\Phi(Z)$. Notice that $P_\Phi(Z)$ contains all periodic irreducible components of Z, but it is possible for $P_\Phi(Z)$ to also contain some non-periodic irreducible components of Z as well. We let $N_\Phi(Z)$ denote the union of the irreducible components of Z that are not contained in $P_\Phi(Z)$.

For each $Z_i \subseteq N_\Phi(Z)$, the sequence $\dim \overline{\Phi^n(Z_i)}$ is weakly decreasing and converges to some $d_i \in \mathbb{N}$ since U has finite Krull dimension. Let

$$\nu_i \coloneqq (d_i, \dim Z_i).$$

We put a strict total order $\prec$ on $(\mathbb{N} \cup \{-\infty\}) \times (\mathbb{N} \cup \{-\infty\})$ by declaring $(a, b) \prec (a', b')$ if $a < a'$, or if $a = a'$ and $b < b'$. The relations $\preceq$, $\succ$, and $\succeq$ are then defined in the natural way.

Definition 3.8. With notation as above, we define the *stable non-periodic dimension* $\nu(Z)$ of Z to be the maximum ν_i with respect to $\prec$. If $N_\Phi(Z)$ is empty, we define $\nu(Z) = (-\infty, -\infty)$.

The following is the main technical result of this subsection.

Proposition 3.9. *Let U be a Noetherian topological space of finite Krull dimension and $\Phi\colon U \longrightarrow U$ a continuous map. Suppose that $T \subseteq \mathbb{N}$ has positive density and $Z \subseteq U$ is a closed subset with $N_\Phi(Z) \neq \varnothing$. Then there exist infinitely many $j \in \Sigma(T)$ with $\nu(Z) \succ \nu(Z \cap \Phi^{-j}(Z))$.*

Proof. Let $Z_1, \ldots, Z_r$ be the irreducible components of Z. After relabelling, we may assume

$$N_\Phi(Z) = Z_1 \cup \cdots \cup Z_s \qquad \text{and} \qquad P_\Phi(Z) = Z_{s+1} \cup \cdots \cup Z_r.$$

Let $L \in \mathbb{Z}^+$ such that $\Phi^L(P_\Phi(Z)) \subseteq P_\Phi(Z)$. By Remark 3.3, there is some $a \in \{0, 1, \ldots, L-1\}$ such that $T \cap (a + L\mathbb{N})$ has positive density. Replacing T by $T \cap (a + L\mathbb{N})$, we can assume that all elements of $\Sigma(T)$ are multiples of L.

Fix m sufficiently large so that

$$\dim \overline{\Phi^m(Z_i)} = \lim_{n \to \infty} \dim \overline{\Phi^n(Z_i)}$$

for $i \leq s$, and let $\nu(Z) = (d, e)$. After relabeling, we may assume that there exist $1 \leq \ell \leq t \leq s$ such that:

(1) $\dim \overline{\Phi^m(Z_i)} = d$ and $\dim Z_i = e$ for $i \le \ell$,
(2) $\dim \overline{\Phi^m(Z_i)} = d$ and $\dim Z_i < e$ for $\ell < i \le t$,
(3) $\dim \overline{\Phi^m(Z_i)} < d$ for $t < i \le s$.

We first claim that for every $j \in \Sigma(T)$, we have

$$P_\Phi(Z \cap \Phi^{-j}(Z)) \supseteq P_\Phi(Z). \tag{3.1}$$

To see this, first note that since j is a multiple of L, we have $Z \cap \Phi^{-j}(Z) \supseteq P_\Phi(Z)$. So, it remains to show that every irreducible component Z_i contained in $P_\Phi(Z)$ is also an irreducible component of $Z \cap \Phi^{-j}(Z)$. Since Z_i is irreducible, it is contained in some irreducible component Z_i' of $Z \cap \Phi^{-j}(Z)$. Then $Z_i \subseteq Z_i' \subseteq Z \cap \Phi^{-j}(Z) \subseteq Z$. As Z_i is already an irreducible component of Z and Z_i' is irreducible, it follows that $Z_i = Z_i'$ is an irreducible component of $Z \cap \Phi^{-j}(Z)$.

By equation (3.1), we necessarily have $\nu(Z \cap \Phi^{-j}(Z)) \preceq \nu(Z)$. Suppose that

$$\nu(Z \cap \Phi^{-j}(Z)) = \nu(Z) \tag{3.2}$$

for every sufficiently large $j \in \Sigma(T)$. We shall derive a contradiction in the remainder of the proof.

We next claim that for every sufficiently large $j \in \Sigma(T)$, there is some $i \le \ell$ such that

$$Z_i \subseteq \Phi^{-j}(Z). \tag{3.3}$$

If $j \in \Sigma(T)$ is sufficiently large, then by equation (3.2), there is an irreducible component C of $Z \cap \Phi^{-j}(Z)$ not contained in $P_\Phi(Z \cap \Phi^{-j}(Z))$ such that $\dim C = e$ and $\dim \overline{\Phi^n(C)} \ge d$ for all $n \ge 0$. We have $C \subseteq Z_i \cap \Phi^{-j}(Z)$ for some $1 \le i \le r$. By equation (3.1), we see $C \not\subseteq P_\Phi(Z)$, and so Z_i is not an irreducible component contained in $P_\Phi(Z)$, i.e., $i \le s$. Next observe that

$$d \le \dim \overline{\Phi^m(C)} \le \dim \overline{\Phi^m(Z_i \cap \Phi^{-j}(Z))} \le \dim \overline{\Phi^m(Z_i)},$$

and so $i \le t$. Moreover, since $C \subseteq Z_i \cap \Phi^{-j}(Z) \subseteq Z_i$, we see $\dim Z_i \ge e$ and hence $i \le \ell$. By dimension contraints, $C = Z_i$ which implies equation (3.3).

Since Proposition 3.6 shows that $\Sigma(T)$ has positive density, by the subadditivity of natural density, there exists a fixed $i \in \{1, \dots, \ell\}$ and a positive density subset $\Sigma_i \subseteq \Sigma(T)$ such that equation (3.3) holds for all $j \in \Sigma_i$. Further refining, there exists $k \in \{1, \dots, r\}$ and a positive density subset $\Sigma_{i,k} \subseteq \Sigma_i$ such that

$$Z_i \subseteq \Phi^{-j}(Z_k)$$

for all $j \in \Sigma_{i,k}$.

We next show that $k \le s$. If this were not the case, then $Z_k \subseteq P_\Phi(Z)$ and so $\Phi^j(Z_i) \subseteq Z_k \subseteq P_\Phi(Z)$. In particular, since j is a multiple of L, we have $\Phi^j(Z_i \cup P_\Phi(Z)) \subseteq P_\Phi(Z) \subseteq Z_i \cup P_\Phi(Z)$. By maximality of $P_\Phi(Z)$, it follows that $P_\Phi(Z) = Z_i \cup P_\Phi(Z)$, and hence $Z_i \subseteq P_\Phi(Z)$, a contradiction.

Since $\Sigma_{i,k}$ is infinite, there exist $a, b \in \Sigma_{i,k}$ with $b - a, a > m$. We write $b = a + Lc$ with $c > 0$. Since $Z_i \subseteq \Phi^{-a}(Z_k)$, we have $\Phi^{a+Lc}(Z_i) = \Phi^{Lc}(\Phi^a(Z_i)) \subseteq \Phi^{Lc}(Z_k)$, and hence $\overline{\Phi^{a+Lc}(Z_i)} \subseteq \overline{\Phi^{Lc}(Z_k)}$. As $a + Lc, Lc > m$, we see $\dim \overline{\Phi^{Lc}(Z_k)} \le d = \dim \overline{\Phi^{a+Lc}(Z_i)}$. Then by irreducibility of Z_k, we have $\overline{\Phi^{a+Lc}(Z_i)} = \overline{\Phi^{Lc}(Z_k)}$. On the other hand, $b \in \Sigma_{i,k}$, so $\Phi^{a+Lc}(Z_i) \subseteq Z_k$, which implies

$$\Phi^{Lc}(Z_k) \subseteq \overline{\Phi^{Lc}(Z_k)} = \overline{\Phi^{a+Lc}(Z_i)} \subseteq Z_k.$$

So, Z_k is periodic and hence contained in $P_\Phi(Z)$, contradicting the fact that $k \le s$. $\square$

Lemma 3.10. *Let U be a non-empty Noetherian topological space of Krull dimension d. Suppose that*

$$Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_m$$

is a descending chain of non-periodic closed subsets of U such that $\nu(Z_0) \succ \nu(Z_1) \succ \cdots \succ \nu(Z_m)$. Then $m < (d+1)^2$.

Proof. We necessarily have $\nu(Z_0) \preceq (d, d)$. Write $\nu(Z_i) = (d_i, e_i)$. Then by definition of $\prec$, we have $d \ge d_0 \ge d_1 \ge \cdots \ge d_m$. For $s \in \{0, 1, \ldots, d\}$, let $A_s = \{i : d_i = s\}$. Then A_s is an interval. Notice that if $A_s = \{j, j+1, \ldots, j+\ell\}$, then since $d_j = \cdots = d_{j+\ell} = s$, we must have $e_j > e_{j+1} > \cdots > e_{j+\ell}$. Since $e_j \le d$, we see that $\ell \le d$. Hence $|A_s| \le d+1$ for each $s \in \{0, 1, 2, \ldots, d\}$. Then since $\{0, 1, 2, \ldots, m\} = A_0 \cup A_1 \cup \cdots \cup A_d$, we see that $m + 1 \le (d+1)^2$, as required. $\square$

3.3. **Finishing the proof of Theorem 1.4.**

Proof of Theorem 1.4. We divide the proof into several steps.

Step 1. We shall start with the following set-up. Let

$$U = X \smallsetminus \bigcup_{n \in \mathbb{N}} (I_{\Phi^n} \cup I_{f \circ \Phi^n}),$$

which may not be open. We endow U with the subspace topology, thereby making it a Noetherian topological space. Clearly, $U(\overline{\mathbb{Q}}) = X_{\Phi,f}(\overline{\mathbb{Q}})$. If U does not have any $\overline{\mathbb{Q}}$-points, then the theorem is vacuously true, so we assume that there is an $x \in U(\overline{\mathbb{Q}})$ such that $f(\mathcal{O}_\Phi(x))$ is infinite. We may also assume that X, Φ, f and x are all defined over a fixed number field K. By construction, $\Phi|_U$ is a regular self-map of U and $f|_U : U \longrightarrow \mathbb{P}^1$ is regular; by abuse of notation, we denote these restriction maps by Φ and f, respectively. Finally, replacing U by the Zariski closure of the orbit $\mathcal{O}_\Phi(x)$ in U, we may assume $\mathcal{O}_\Phi(x)$ is Zariski dense in U; see also the end of the introduction of [BGS].

So, from now on, we may assume:

(1) $U \subseteq X$ is a Noetherian topological space;
(2) Φ is regular on U and $\Phi(U) \subseteq U$;
(3) $f : U \longrightarrow \mathbb{P}^1$ is regular;
(4) $x \in U(K)$ and $\mathcal{O}_\Phi(x)$ is Zariski dense in U;
(5) f is non-constant on each periodic component of the Zariski closure of $\mathcal{O}_\Phi(x)$.

Step 2. Let $T \subseteq \mathbb{N}$ be a subset of positive density, $d = \dim(U \times U)$, and

$$Z_0 = \{(u, v) \in U \times U : f(u) = f(v)\}.$$

Note that Z_0 is a closed subset of $U \times U$ since it is the inverse image of the diagonal $\Delta_{\mathbb{P}^1} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ under the product map $(f, f): U \times U \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Applying Proposition 3.9 to $Z_0 \subset U \times U$, the product map (Φ, Φ), and T, we see that there is some $i_0 \in \Sigma(T)$ such that $T_1 := T \cap (T + i_0)$ has positive density and $Z_1 := Z_0 \cap (\Phi, \Phi)^{-i_0}(Z_0)$ satisfies $\nu(Z_1) \prec \nu(Z_0)$. If $Z_1 = P_{(\Phi,\Phi)}(Z_1)$ is periodic under (Φ, Φ), then let $m = 1$. Otherwise, applying Proposition 3.9 to Z_1 yields an element $i_1 \in \Sigma(T_1) \subseteq \Sigma(T)$ with $i_0 < i_1$ such that $T_2 := T_1 \cap (T_1 + i_1)$ has positive density and $Z_2 := Z_1 \cap (\Phi, \Phi)^{-i_1}(Z_1)$ satisfies $\nu(Z_2) \prec \nu(Z_1)$. Proceeding in this manner, we obtain a sequence of integers

$$0 < i_0 < i_1 < \cdots < i_m$$

and a descending chain of closed subsets $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_m$ such that $\nu(Z_i) \succ \nu(Z_{i+1})$ and $Z_m = P_{(\Phi,\Phi)}(Z_m)$, i.e., Z_m is periodic. Furthermore, by construction, if

$$S := \left\{ \sum_{i \in I} i : I \subseteq \{i_0, \ldots, i_m\} \right\}$$

then

$$T' := \bigcap_{s \in S} (T + s) \subseteq T$$

has positive density,

$$Z_m = \bigcap_{s \in S} (\Phi, \Phi)^{-s}(Z_0)$$

and there is some $L \in \mathbb{Z}^+$ such that

$$(\Phi, \Phi)^L(Z_m) \subseteq Z_m.$$

Lastly, Lemma 3.10 implies

$$|S| \leq 2^{m+1} \leq 2^{(d+1)^2}.$$

Notice that $Z_m \neq \varnothing$ since the diagonal $\Delta_U \subseteq U \times U$ is contained in Z_0 and $(\Phi, \Phi)^{-n}(\Delta_U) \supseteq \Delta_U$ for every $n \geq 0$.

Step 3. By Schanuel's Theorem 2.1, there exists a positive real number $\kappa > 0$ depending only on the number field K such that for all sufficiently large B, we have

$$\#\big\{y \in \mathbb{P}^1(K) : h(y) \leq \log B\big\} \leq B^\kappa. \tag{3.4}$$

Choose an ϵ independent of T such that $0 < \epsilon < (2^{(d+1)^2+1}\kappa)^{-1}$. We shall prove that this choice of ϵ satisfies the conclusion of Theorem 1.4. Suppose to the contrary that

$$\limsup_{n \in T} \frac{h(f(\Phi^n(x)))}{\log n} \leq \epsilon.$$

In particular, there is a positive integer N_0 such that $h(f(\Phi^n(x))) \leq 2\epsilon \log n$ for all $n \in T$ with $n \geq N_0$.

First, by equation (3.4), we have

$$\#\left\{f(\Phi^n(x)) \in \mathbb{P}^1(K) : n \in T,\ N_0 \leq n \leq N\right\} \leq N^{2\epsilon\kappa} \tag{3.5}$$

for N sufficiently large. Let

$$T'' = \bigcap_{s \in S}(T - s) \subseteq T.$$

Since $\overline{d}(T') > 0$ and $T'' = T' - (i_0 + \cdots + i_m)$, we see $\overline{d}(T'') > 0$. By construction, for any $j \in T''$, we have $j + s \in T$ for every $s \in S$. In particular, equation (3.5) implies

$$\begin{aligned}
&\#\left\{\left(f(\Phi^{j+s}(x))\right)_{s\in S} \in \mathbb{P}^1(K)^{|S|} : j \in T'',\ N_0 \leq j \leq N - (i_0 + \cdots + i_m)\right\} \\
&\leq \prod_{s \in S} \#\left\{f(\Phi^{j+s}(x)) \in \mathbb{P}^1(K) : j \in T'',\ N_0 - s \leq j \leq N - s\right\} \\
&\leq \prod_{s \in S} \#\left\{f(\Phi^{j+s}(x)) \in \mathbb{P}^1(K) : j + s \in T,\ N_0 \leq j + s \leq N\right\} \\
&\leq N^{2\epsilon\kappa \cdot |S|}.
\end{aligned} \tag{3.6}$$

Step 4. Let L be as in Step 2. Since $\overline{d}(T'' \cap [N_0, \infty)) = \overline{d}(T'') > 0$, by subadditivity of natural density, there exists an integer $a \in [0, L)$ such that $\{j \in T'' : j \geq N_0,\ j \equiv a \pmod{L}\}$ has positive density. Then by the definition of natural density (see Definition 3.1), there exists a subsequence $(n_\ell)_{\ell \in \mathbb{Z}^+}$ of positive integers and a positive real number $\delta > 0$, such that

$$\#\left\{j \in T'' : N_0 \leq j \leq n_\ell,\ j \equiv a \pmod{L}\right\} \geq \delta n_\ell$$

for sufficiently large ℓ. Now, replacing δ by a smaller positive number if necessary, we can further assume that

$$\#\left\{j \in T'' : N_0 \leq j \leq n_\ell - (i_0 + \cdots + i_m),\ j \equiv a \pmod{L}\right\} \geq \delta n_\ell, \tag{3.7}$$

for sufficiently large ℓ.

Recall from Step 2 that $|S| \leq 2^{m+1} \leq 2^{(d+1)^2}$ and hence $2\epsilon\kappa \cdot |S| < 1$. Therefore, we can choose ℓ large enough such that $\delta n_\ell > n_\ell^{2\epsilon\kappa \cdot |S|}$. Combining equation (3.7) with (3.6) where $N = n_\ell$, a direct counting argument yields that there exist positive integers $i, j \in T''$ with $i < j$ such that

$$f(\Phi^{i+s}(x)) = f(\Phi^{j+s}(x)) \text{ for all } s \in S \quad \text{and} \quad i \equiv j \pmod{L}.$$

Then by definition, $(\Phi^i(x), \Phi^j(x)) \in Z_m$ (see Step 2 for the construction and properties of Z_m). Since $(\Phi, \Phi)^L(Z_m) \subseteq Z_m$, we have $(\Phi^{kL+i}(x), \Phi^{kL+j}(x)) \in Z_m$ for every $k \in \mathbb{N}$. As $Z_m \subseteq Z_0$, the definition of Z_0 yields

$$f(\Phi^{kL+i}(x)) = f(\Phi^{kL+j}(x)).$$

It thus follows from the fact that $i \equiv j \pmod{L}$ that the sequence $(f(\Phi^{kL}(\Phi^i(x))))_{k \in \mathbb{N}}$ is periodic. In particular, the orbit $\mathcal{O}_{\Phi^L}(\Phi^i(x))$ of $\Phi^i(x)$ under Φ^L is contained in finitely many fibers $F_1, \dots, F_s$ of f. Note that

$$\mathcal{O}_\Phi(\Phi^i(x)) = \bigcup_{t=0}^{L-1} \mathcal{O}_{\Phi^L}(\Phi^{i+t}(x)) = \bigcup_{t=0}^{L-1} \Phi^t(\mathcal{O}_{\Phi^L}(\Phi^i(x))) \subseteq \bigcup_{t=0}^{L-1} \Phi^t(F_1 \cup \dots \cup F_s).$$

It thus follows that

$$\overline{\mathcal{O}_\Phi(\Phi^i(x))} \subseteq \bigcup_{t=0}^{L-1} \overline{\Phi^t(F_1)} \cup \dots \cup \overline{\Phi^t(F_s)}.$$

Since the full forward orbit $\mathcal{O}_\Phi(x)$ is Zariski dense in U, the fibers F_i are periodic components of U, which contradicts our assumption in Step 1 that f is non-constant on each periodic component of the Zariski closure of $\mathcal{O}_\Phi(x)$. We thus complete the proof of Theorem 1.4. □

Remark 3.11. As mentioned in Remark 1.6, our proofs of Theorems 1.2, 1.4, and 1.5 are characteristic-free except for our use of Schanuel's Theorem 2.1 for number fields (e.g., in Step 3 of the proof of Theorem 1.4). We outline below an extension of our main results to function fields of higher transcendence degree over finite fields.

Let Y be an irreducible projective variety in $\mathbb{P}^m$, regular in codimension one, defined over a finite field $\mathbb{F}_q$. Let K be the function field $\mathbb{F}_q(Y)$ of Y. If $P = [f_0 : \ldots : f_n]$ is a K-rational point of $\mathbb{P}^n$, then the height function h_K can be equivalently defined by

$$h_K(P) \coloneqq \deg \phi^* \mathcal{O}_{\mathbb{P}^n}(1), \tag{3.8}$$

where ϕ is an $\mathbb{F}_q$-rational map determined by P as follows:

$$\phi \colon Y \dashrightarrow \mathbb{P}^n, \quad y \in Y(\mathbb{F}_q) \mapsto [f_0(y) : \ldots : f_n(y)] \in \mathbb{P}^n(\mathbb{F}_q).$$

Note that this definition of height is dependent upon the embedding $Y \hookrightarrow \mathbb{P}^m$, but given two height functions h_1, h_2 on $\mathbb{P}^n_K$ defined in this way, there are constants $C_1, C_2 > 0$ such that

$$C_1 h_1(P) \leq h_2(P) \leq C_2 h_1(P)$$

for all $P \in \mathbb{P}^n(K)$. See, for instance, [Lan83, §3.3], [HS00, §B.10] or [BG06, §2.4] for a brief account of the theory of heights over function fields.

Let d denote the transcendence degree of K over $\mathbb{F}_q$ (i.e., $d = \dim Y$). By [Har77, Theorem I.7.5], there is a unique polynomial $p_Y(z) \in \mathbb{Q}[z]$ of degree d such that for all k sufficiently large, $p_Y(k) = \dim S(Y)_k$, the degree k homogeneous piece of the homogeneous coordinate ring $S(Y)$ of Y with respect to the embedding into $\mathbb{P}^m$. Then using the notation of equation (3.8), the number of $P \in \mathbb{P}^n(K)$ for which $\deg \phi^* \mathcal{O}_{\mathbb{P}^n}(1) = k$ is at most the number of $(n+1)$-tuples of degree k elements in $S(Y)$, which is $q^{(n+1)p_Y(k)}$ for k sufficiently large. It follows that there is a constant $C = C(Y) > 0$ such that for $k \gg 0$, the number of $P \in \mathbb{P}^n(K)$ with $h_Y(P) \leq k$ is at most $q^{C(n+1)k^d}$. On the other hand, if we fix a nonzero element $g_0 \in S(Y)_k$ and let $g_1, \dots, g_n$ vary over elements of $S(Y)_k$, we see that the elements $[1 : g_1/g_0 : \ldots : g_n/g_0] \in \mathbb{P}^n(K)$ give rise to $q^{np_Y(k)}$ distinct elements of height at most k in $\mathbb{P}^n(K)$. Thus, similarly, for $k \gg 0$ the number of elements of height at most k in $\mathbb{P}^n(K)$ is at least $q^{C'nk^d}$ for some positive constant C'. In summary, there are positive constants C_1 and C_2, depending only upon Y and q, such that

$$B^{C_1 n} \leq \#\{P \in \mathbb{P}^n(K) : h_K(P) \leq (\log B)^{1/d}\} \leq B^{C_2(n+1)} \tag{3.9}$$

for all B sufficiently large. In fact, we only require the upper bound in our arguments.

Thus, replacing our use of Schanuel's Theorem with (3.9), we find that when K is a function field of transcendence degree d over a finite field $\mathbb{F}_q$, analogues of Theorems 1.2, 1.4, and 1.5 hold with $\log n$ replaced by $(\log n)^{1/d}$. Moreover, this gap is optimal in terms of what can be attained via our methods. In particular, in the global case (i.e., when $d = 1$), the exact same conclusion holds as in the number field case.

4. Applications of Theorems 1.2 and 1.4

4.1. **Weak** $\liminf$ **Height gap.** Theorem 1.5, which asserts that Conjecture 1.3 holds away from a set of density zero, is an immediate consequence of Theorem 1.4.

Proof of Theorem 1.5. Let $\epsilon > 0$ be the positive real number as in Theorem 1.4 and

$$S := \left\{ n \in \mathbb{N} : \frac{h(f(\Phi^n(x)))}{\log n} \leq \frac{\epsilon}{2} \right\}.$$

Then by Theorem 1.4, S has density zero, which concludes the proof. $\square$

Remark 4.1. One can see from the proof of Theorem 1.4 (in particular, Step 3) that our constants ϵ and C depend only on the fixed number field K and the dimension of X; moreover, they are decreasing as the dimension of X increases.

4.2. **Height gaps for D-finite power series.** We apply Theorem 1.2 to obtain a simple proof of Theorem 1.9 recovering the univariate case of [BNZ, Theorem 1.3(c)].

Proof of Theorem 1.9. If $\sum_{n\geq 0} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is a D-finite power series, then there is a rational self-map $\Phi\colon \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ for some $d \geq 2$, a point $c \in \mathbb{P}^d(\overline{\mathbb{Q}})$, and a rational map $f\colon \mathbb{P}^d \dashrightarrow \mathbb{P}^1$ such that $a_n = f(\Phi^n(c))$ for $n \gg 0$; see, e.g., [BGT16, §3.2.1]. So, Theorem 1.2 immediately implies Theorem 1.9. $\square$

4.3. **A weak dynamical Mordell–Lang theorem.** As mentioned before in the introduction, the $\liminf$ Height Gap Conjecture 1.3 would imply the Dynamical Mordell–Lang Conjecture. Similarly, we deduce Theorem 1.10 as an application of Theorem 1.5.

Proof of Theorem 1.10. For any $n \in \mathbb{N}$, we denote by $Z_{\geq n}$ the Zariski closure of $\{\Phi^i(x) : i \geq n\}$ in X. Since X is a Noetherian topological space, there is some $m \in \mathbb{N}$ such that $Z_{\geq n} = Z_{\geq m}$ for every $n \geq m$. Denote $Z_{\geq m}$ by Z and $\Phi^m(x)$ by x_1. It then suffices to prove Theorem 1.10 for $(Z, \Phi|_Z, x_1, Y \cap Z)$.

Let $Z_1, \ldots, Z_r$ denote the irreducible components of Z and let $Y_i := Y \cap Z_i$. Then $x_1 \in Z_i$ for some i. After relabeling, we may assume that $x_1 \in Z_1$. For each $i = 2, \ldots, r$, choose an arbitrary $x_i \in \mathcal{O}_\Phi(x_1) \cap Z_i$; the intersection is non-empty since $\mathcal{O}_\Phi(x_1)$ is dense in Z by definition. We claim that $\Phi|_Z$ cyclically permutes the irreducible components Z_i of Z. To see this, first note that $\Phi|_Z$ is a dominant rational self-map of Z, so it permutes the Z_i. Suppose that $(Z_{i_1}, \dots, Z_{i_s})$ is an s-cycle under $\Phi|_Z$ with $1 \le s \le r$, and consider the forward orbit $\mathcal{O}_\Phi(x_{i_1})$ of $x_{i_1} \in Z_{i_1}$. Clearly, the closure of $\mathcal{O}_\Phi(x_{i_1})$ in X is contained in the union of $Z_{i_1}, \dots, Z_{i_s}$. On the other hand, $x_{i_1} = \Phi^n(x)$ for some $n \ge m$, and so the closure of $\mathcal{O}_\Phi(x_{i_1})$ in X is $Z_{\ge n} = Z$. It follows that $s = r$ and hence $\Phi|_Z$ is a cyclic permutation $(Z_{i_1}, \dots, Z_{i_r})$. Hence $\Phi^r(Z_i) \subseteq Z_i$ for each i. Moreover, after relabeling, we may assume that $\Phi(Z_i) \subseteq Z_{i+1}$ for $i = 1, \dots, r-1$ and $\Phi(Z_r) \subseteq Z_1$. So, for $i = 2, \dots, r$, our x_i could be taken to be $\Phi^{i-1}(x_1)$. Therefore, by the subadditivity of natural density, it suffices to show Theorem 1.10 for $(Z_i, \Phi^r|_{Z_i}, x_i, Y_i)$ for each $i = 1, \dots, r$.

We claim further that for each i, the forward orbit $\mathcal{O}_{\Phi^r}(x_i)$ of $x_i \in Z_i$ under Φ^r is dense in Z_i. In fact, if we denote the irreducible decomposition of the closure of $\mathcal{O}_{\Phi^r}(x_i)$ by $W_{i,1}, \dots, W_{i,r_i}$, then

$$\bigcup_{i=1}^r Z_i = Z = \overline{\mathcal{O}_\Phi(x_1)} = \bigcup_{i=1}^r \overline{\mathcal{O}_{\Phi^r}(x_i)} = \bigcup_{i=1}^r \bigcup_{j=1}^{r_i} W_{i,j}.$$

Since Z_i is irreducible, $Z_i \subseteq W_{k,j}$ for some $1 \le k \le r$ and $1 \le j \le r_k$. However, we note that $W_{k,j} \subseteq \overline{\mathcal{O}_{\Phi^r}(x_k)} \subseteq Z_k$. As $Z_1, \dots, Z_r$ are the irreducible components of Z, we must have $k = i$. The claim $\overline{\mathcal{O}_{\Phi^r}(x_i)} = Z_i$ thus follows.

We shall prove that either $\mathcal{O}_{\Phi^r}(x_i) \subseteq Y_i$, or the set

$$A_i \coloneqq \left\{ n \in \mathbb{N} : \Phi^{rn}(x_i) \in Y_i \right\}$$

has density zero, thereby proving Theorem 1.10 for $(Z_i, \Phi^r|_{Z_i}, x_i, Y_i)$. If $Y_i = Z_i$ or $Y_i = \varnothing$, then the result is immediate. Thus we may assume, without loss of generality, that Y_i is a non-empty proper subvariety of Z_i. We pick a non-constant morphism $f_i \colon Z_i \longrightarrow \mathbb{P}^1$ such that $f_i(Y_i) = 1$; one can accomplish this by choosing a non-constant rational function F_i vanishing on Y_i and then letting $f_i \coloneqq F_i + 1$. In particular, if $\Phi^{rn}(x_i) \in Y_i$, then $h(f_i(\Phi^{rn}(x_i))) = 0$. On the other hand, as $\mathcal{O}_{\Phi^r}(x_i)$ is dense in Z_i, it follows from Theorem 1.5 that there exist a positive constant C and a set $S \subset \mathbb{N}$ of zero density such that for any $n \in \mathbb{N} \setminus S$, the height of $f_i(\Phi^{rn}(x_i))$ is greater than $C \log n > 0$; in particular, for such an n, $\Phi^{rn}(x_i) \notin Y_i$. It follows that $A_i \subseteq S$ has density zero, as required. We hence complete the proof of Theorem 1.10. $\square$

5. Weak lim inf Height Gap for multiple maps: Proof of Theorem 1.11

We begin this section by proving Theorem 1.11.

Proof of Theorem 1.11. As before, we first fix a number field K so that X, $\Phi_1, \dots, \Phi_m$, f, and x are all defined over K. We shall prove the theorem by strong induction on the number m of commuting self-maps $\Phi_1, \dots, \Phi_m$ for any irreducible quasi-projective variety X and any non-constant rational function f defined over the above K. The base case $m = 1$ has been handled in Theorem 1.5 with a constant $C_{K,X} > 0$ depending only on the fixed number field K and the dimension of X (see Remark 4.1). So, let us suppose that our theorem holds true for any $k < m$ commuting self-maps of any irreducible quasi-projective variety W and for any non-constant rational function on W defined over K, with a constant $C_{K,W} > 0$ depending only on K and the dimension of W.

Let G denote the semigroup of rational self-maps $\Phi^{\mathbf{n}} = \Phi_1^{n_1} \circ \cdots \circ \Phi_m^{n_m}$ of X generated by $\Phi_1, \ldots, \Phi_m$. By assumption, the forward orbit $\mathcal{O}_{\Phi_1,\ldots,\Phi_m}(x)$ of $x \in X_{\Phi_1,\ldots,\Phi_m,f}(\overline{\mathbb{Q}})$ under G is Zariski dense in X. Without loss of generality, we may assume that for any $\mathbf{n}' \in \mathbb{N}^m$, the forward orbit $\mathcal{O}_{\Phi_1,\ldots,\Phi_m}(x')$ of $x' := \Phi^{\mathbf{n}'}(x) = \Phi_1^{n_1'} \circ \cdots \circ \Phi_m^{n_m'}$ under G is also Zariski dense in X (for the purpose of induction). Otherwise, there must be some fixed i with $1 \le i \le m$ and some fixed j with $0 \le j < n_i'$ such that the forward orbit of $\Phi_i^j(x)$ under the $m-1$ commuting self-maps $\Phi_1, \ldots, \Phi_{i-1}, \Phi_{i+1}, \ldots, \Phi_m$ is Zariski dense in X; Theorem 1.11 thus follows from the induction hypothesis with the same positive constant $C_{K,X}$.

As before, let Z_{n_m} denote the Zariski closure of $\{\Phi_m^k(x) : k \ge n_m\}$ in X. We may pick an n_m such that $Z := Z_{n_m} = Z_{n_m+1} = \cdots$. By the above remark, we may replace x by $\Phi_m^{n_m}(x)$ and assume that the orbit closure of x under Φ_m is equal to Z and that Φ_m permutes the irreducible components of Z. In particular, there is some fixed $L \in \mathbb{Z}^+$ such that the Zariski closure of the orbit of x under Φ_m^L is an irreducible component Z_1 of Z. For each $i \in \{0, \ldots, L-1\}$, let R_i denote the subset $\{n \in \mathbb{N} : n \equiv i \pmod{L}\}$ of $\mathbb{N}$. Now, by the subadditivity of natural density, it suffices to prove the following

Claim A$_i$. For each $i \in \{0, \ldots, L-1\}$, the subset

$$S_i := \left\{ n \in R_i : h(f(\Phi^{\mathbf{n}}(x))) \le C \log \|\mathbf{n}\|_1 \text{ for every } \mathbf{n} \in \mathbb{N}^m \text{ with } \|\mathbf{n}\|_1 = n \right\}$$

of R_i has density zero, where $C := C_{K,X} > 0$ is the same constant as above.

In fact, the zero density set $S := \cup S_i$ and the above $C > 0$ satisfy the conclusion of Theorem 1.11. Let us fix an $i \in \{0, \ldots, L-1\}$. If there exists an element $\Psi = \Phi^{\mathbf{a}} \in G$ with $\|\mathbf{a}\|_1 \in R_i$ such that f is non-constant on $\Psi(Z_1)$, then by applying Theorem 1.5 to the rational self-map Φ_m^L of $\Psi(Z_1)$, the non-constant rational function $f|_{\Psi(Z_1)}$, and the K-rational point $\Psi(x)$ of $\Psi(Z_1)$, there exists a constant $C_{K,\Psi(Z_1)} > 0$ such that

$$S' := \left\{ n \in \mathbb{N} : h(f(\Phi_m^{Ln} \circ \Psi(x))) \le C_{K,\Psi(Z_1)} \log(Ln + \|\mathbf{a}\|_1) \right\}$$

has density zero. Now, for any $s \in S_i \subset R_i$, without loss of generality, we may assume that $s \ge \|\mathbf{a}\|_1$ and write $s = Ln + \|\mathbf{a}\|_1$ for some $n \in \mathbb{N}$, since $\|\mathbf{a}\|_1 \in R_i$. By the definitions of S_i and S', as well as the fact that $C_{K,X} \le C_{K,\Psi(Z_1)}$ (see Remark 4.1), we have $n \in S'$. This yields that $S_i \subset LS' + \|\mathbf{a}\|_1$ up to finitely many elements, while the latter clearly has density zero. Hence Claim A$_i$ follows. So, from now on, let us consider the case when f is constant on $\Psi(Z_1)$ for every $\Psi = \Phi^{\mathbf{n}} \in G$ with $\|\mathbf{n}\|_1 \in R_i$.

For the fixed i, let E denote the subfield of the function field $K(X)$ of X generated by all rational functions $f \circ \Phi^{\mathbf{n}}$ with $\|\mathbf{n}\|_1 \in R_i$. Since $K(X)$ is finitely generated over K, so is E. We may then assume that there exist $\mathbf{k}_1, \ldots, \mathbf{k}_e \in \mathbb{N}^m$ with each $\|\mathbf{k}_j\|_1 \in R_i$

such that E is generated by $f \circ \Phi^{\mathbf{k}_1}, \dots, f \circ \Phi^{\mathbf{k}_e}$ over K. (Notice that the $f \circ \Phi^{\mathbf{k}_j}$ do not necessarily form a K-basis.) In particular, we have a K-rational map

$$\chi\colon X \dashrightarrow \mathbb{A}^e, \quad x \mapsto (f(\Phi^{\mathbf{k}_1}(x)), \dots, f(\Phi^{\mathbf{k}_e}(x))).$$

Furthermore, if we let Y denote the proper transform of X under χ, then each $\Phi^{\boldsymbol{\ell}} \in G$ with $\|\boldsymbol{\ell}\|_1 \in R_0$ induces a natural rational self-map Π of Y such that $\Pi \circ \chi = \chi \circ \Phi^{\boldsymbol{\ell}}$. Indeed, for each j with $1 \le j \le e$, the rational function $f \circ (\Phi^{\mathbf{k}_j} \circ \Phi^{\boldsymbol{\ell}})$ is in the field E as $\|\mathbf{k}_j + \boldsymbol{\ell}\|_1 = \|\mathbf{k}_j\|_1 + \|\boldsymbol{\ell}\|_1 \in R_i$. Hence, there is a rational function $g_j \in K(T_1, \dots, T_e)$ such that $f \circ (\Phi^{\mathbf{k}_j} \circ \Phi^{\boldsymbol{\ell}}) = g_j(f \circ \Phi^{\mathbf{k}_1}, \dots, f \circ \Phi^{\mathbf{k}_e})$. Clearly, the restriction of the rational self-map of $\mathbb{A}^e$ defined by the g_j to Y is our Π.

Since Y is the proper transform of X under χ, Y is irreducible. Also, since the orbit of x under G is dense in X, it must have dense orbit under the subsemigroup generated by $\Phi_1^L, \dots, \Phi_m^L$. Let Π_j denote the induced self-map of Y from Φ_j^L. Then $y := \chi(x)$ has Zariski dense orbit under $\Pi_1, \dots, \Pi_m$. Recall that by construction the irreducible variety Z_1 is the orbit closure of x under Φ_m^L and f is constant on $\Phi^{\mathbf{n}}(Z_1)$ for every $\mathbf{n} \in \mathbb{N}^m$ with $\|\mathbf{n}\|_1 \in R_i$. It follows that $\chi(Z_1) = y$ is a single point fixed by Π_m. Therefore, if we let H denote the semigroup generated by $\Pi_1, \dots, \Pi_{m-1}$, then the orbit of y under H is Zariski dense in Y and one can deduce from $\chi(\Phi_1^{Ln_1} \circ \dots \circ \Phi_m^{Ln_m}(x)) = \Pi_1^{n_1} \circ \dots \circ \Pi_m^{n_m}(y)$ that

$$f(\Phi_1^{Ln_1} \circ \dots \circ \Phi_m^{Ln_m}(\Phi^{\mathbf{k}_1}(x))) = p_1(\Pi_1^{n_1} \circ \dots \circ \Pi_m^{n_m}(y)) = p_1(\Pi_1^{n_1} \circ \dots \circ \Pi_{m-1}^{n_{m-1}}(y)),$$

where p_1 is the projection of $\mathbb{A}^e$ to the first coordinate. Applying the induction hypothesis to the irreducible variety Y, the semigroup generated by $\Pi_1, \dots, \Pi_{m-1}$, and the projection p_1, we prove Claim A_i by the same argument as before. We hence complete the proof of Theorem 1.11 by induction. □

The example below explains why we consider sets $S \subset \mathbb{N}$ of zero density in Theorem 1.11, rather than subsets $\mathbf{S} \subseteq \mathbb{N}^m$ of zero density,[1] as well as the necessity of the maximum over $\mathbf{n} \in \mathbb{N}^m$ with $\|\mathbf{n}\|_1 = n$.

Example 5.1. We define two self-maps Φ_1 and Φ_2 of $X = \mathbb{A}^1$ as follows:

$$\Phi_1(x) = 2x \quad \text{and} \quad \Phi_2(x) = 0.$$

Let f be the identity map of $\mathbb{A}^1$. It follows that if $\mathbf{T} \subseteq \mathbb{N}^2$, then

$$\limsup_{(n_1,n_2)\in\mathbf{T}} \frac{h(f(\Phi^{n_1,n_2}(1)))}{\log(n_1+n_2)} > 0$$

only if $\mathbf{T}$ contains infinitely many points from the ray $R := \{(n_1, n_2) \in \mathbb{N}^2 : n_2 = 0\}$ which has density zero. On the other hand, the lim inf of the above quantity is zero whenever $\mathbf{T}$ has positive density. Moreover, the only set $\mathbf{T}$ over which the lim inf is positive is a subset of the above ray R plus finitely many points.

[1]Generalizing Definition 3.1, the *upper asymptotic (or natural) density* of $\mathbf{T} \subseteq \mathbb{N}^m$ is defined by $\overline{d}(\mathbf{T}) := \limsup_{n\to\infty} |\mathbf{T} \cap [0, n]^m|/(n+1)^m$.

On the other hand, without the maximum over $\mathbf{n} \in \mathbb{N}^m$ with $\|\mathbf{n}\|_1 = n$ in Theorem 1.11, for any $T \subset \mathbb{N}$, the $\liminf$ over all $\mathbf{n} \in \mathbb{N}^m$ with $\|\mathbf{n}\|_1 \in T$ is zero.

At the end of the introduction, we mention that it appears to be a subtle issue to deduce a multivariate D-finiteness result (e.g., [BNZ, Theorem 1.3(c)]) from our Theorem 1.11. See Lipshitz [Lip89] for the precise definition and basic properties of multivariate D-finite power series. In the univariate case the coefficients of a D-finite power series arise as $f(\Phi^n(c))$ for certain choices of X, Φ, f, and c; see [BGT16, §3.2.1]. However, in Example 5.2 below, we construct a rational function in two variables whose coefficients never arise as $f(\Phi_1^{n_1} \circ \Phi_2^{n_2}(c))$ for any choices of X, Φ_1, Φ_2, f, and c. It is well known that all algebraic functions are D-finite (see [Lip89, Proposition 2.3]).

Example 5.2. Let us consider the following rational function

$$F(z_1, z_2) := \frac{1}{(1 - z_1 z_2)(1 - z_1)} = \sum_{n_2 \geq 0} \sum_{n_1 \geq n_2} z_1^{n_1} z_2^{n_2}.$$

By letting $a_{n_1,n_2} = 1$ if $n_1 \geq n_2 \geq 0$ and $a_{n_1,n_2} = 0$ if $n_2 > n_1 \geq 0$, we may write

$$F(z_1, z_2) = \sum_{n_1, n_2 \in \mathbb{N}} a_{n_1,n_2} z_1^{n_1} z_2^{n_2}.$$

One can show that there is no choice of algebraic variety X, commuting rational self-maps $\Phi_1, \Phi_2 \colon X \dashrightarrow X$, and rational function $f \colon X \dashrightarrow \mathbb{P}^1$ all defined over $\overline{\mathbb{Q}}$, and a point $c \in X_{\Phi_1,\Phi_2,f}(\overline{\mathbb{Q}})$ such that

$$f(\Phi_1^{n_1} \circ \Phi_2^{n_2}(c)) = a_{n_1,n_2},$$

for sufficiently large n_1 and n_2.

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