Algebraic curves:

Let $P(z, w)$ be a non-constant polynomial in 2 cpx variables. Then,

$$C := \{(z, w) \mid P(z, w) = 0\} \subset \mathbb{C}^2$$
$$= \left(\begin{array}{c}\text{algebraic curve}\\ \text{defined by } P\end{array}\right)$$

The algebraic curve is said to be <u>smooth</u> at the pt (z_0, w_0) if $\nabla P := \left(\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w}\right) \neq 0$ at (z_0, w_0).
Otherwise, it is singular at (z_0, w_0).
[Here, $\frac{\partial P}{\partial z}$ & $\frac{\partial P}{\partial w}$ are defined as in the real case, so that $\frac{\partial P}{\partial z}$ & $\frac{\partial P}{\partial w}$ are again polynomials.]

E.g.: 1) $P(z, w) = w^2 - z \rightsquigarrow \nabla P = (2w, -1) \neq (0,0)$ everywhere.

$\Longrightarrow C := \{z = w^2\}$ is smooth everywhere.

2) $P(z, w) = w^2 - z^3 \rightsquigarrow \nabla P = (2w, -3z^2) = (0, 0)$
$\Longleftrightarrow (z, w) = (0,0)$

$\Longrightarrow C := \{z = w^2\}$ is singular at $(0,0)$, but smooth everywhere else.

[PROP: Let $S = C \setminus \{\text{sing. pts}\}$. Then S admits a natural cpx str., making it into a Riemann surface (or at least its conn. comp.)

This proposition is a direct consequence of the Implicit Function Thm.

(B)

Thm: (Implicit Fct Thm).

Suppose (z_0, w_0) is a pt in $\mathbb{C}$ such that $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$. Then, $\exists$ disc D_1 centered at z_0 in $\mathbb{C}$, a disc D_2 centered at w_0 in $\mathbb{C}$, and a holomorphic map $\phi : D_1 \subseteq \mathbb{C} \to D_2 \subseteq \mathbb{C}$ with $\phi(z_0) = w_0$ such that

$$X \cap (D_1 \times D_2) = \{(z, \phi(z)) : z \in D_1\}.$$

Pf: We will need the following:

CLAIM: Let g be a holomorphic fct on an open set containing a disc D, such that g does not vanish on ∂D. Then,

$$\frac{1}{2\pi i}\int_{\partial D} \frac{g'(w)}{g(w)}\, dw = \left(\begin{array}{c}\text{\# of zeroes} \\ \text{of } g \text{ in } D \\ \text{(counted with} \\ \text{multiplicity)}\end{array}\right)$$

Moreover, if g has only one zero in D, say w_1, then

$$w_1 = \frac{1}{2\pi i}\int_{\partial D} \frac{w\, g'(w)}{g(w)}\, dw.$$

Pf of claim: This follows from the Residue Thm. By the Residue Thm, if h hol. on an open set containing D except possibly at a finite nb of pts $w_1, \dots, w_r$ where it has poles, then

$$\frac{1}{2\pi i}\int_{\partial D} h(w)\, dw = \sum_{i=1}^{r} \operatorname{Res}_{w=w_i} h(w).$$

(c)

Moreover, if h has a pole of multiplicity m at w_i, then

$$\operatorname*{Res}_{w=w_i} h(w) = \lim_{w \to w_i} (w-w_i)^m h(w).$$

Now, suppose that g has a zero of multiplicity l_i at w_i, then

$$g(w) = (w-w_i)^{l_i} \tilde{g}(w)$$

with $\tilde{g}$ analytic and non-zero at w_i. Then,

$$\begin{aligned} g'(w) &= l(w-w_i)^{l_i-1}\tilde{g}(w) + (w-w_i)^{l_i}\tilde{g}'(w) \\ &= (w-w_i)^{l_i-1}\left[l_i\tilde{g}(w) + (w-w_i)\tilde{g}'(w)\right] \end{aligned}$$

and

$$\frac{g'(w)}{g(w)} = \frac{l\tilde{g}(w) + (w-w_i)\tilde{g}'(w)}{(w-w_i)\tilde{g}(w)},$$

which has a simple pole at w_i since $\tilde{g}(w_i) \neq 0$

Then, $$\operatorname*{Res}_{w=w_i} \frac{g'(w)}{g(w)} = \lim_{w\to w_i} (w-w_i)\frac{g'(w)}{g(w)}$$

$$= \lim_{w \to w_i} \frac{l_i\tilde{g}(w) + (w-w_i)\tilde{g}'(w)}{\tilde{g}(w)} = l_i.$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\partial D} \frac{g'(w)}{g(w)}\,dw = \sum_{i=1}^{r} l_i = \left(\begin{array}{c}\#\text{ of zeroes} \\ \text{of } g \text{ in } D \\ \text{counting mult.}\end{array}\right)$$

If, in addition, g has only one zero in D, then $wg'(w)/g(w)$ has only one pole in D and it is

...simple, so that

$$\frac{1}{2\pi i}\int_{\partial D} w\,\frac{g'(w)}{g(w)}\,dw = \operatorname{Res}_{w=w_1} \frac{w\,g'(w)}{g(w)}$$

$$= \lim_{w\to w_1} \underbrace{w}_{\to\, w_1}\; \underbrace{(w-w_1)\frac{g'(w)}{g(w)}}_{\to\, 1} = w_1$$

∎

Let us now consider the following family of fcts of the variable w: $f_z(w) := P(z,w)$, with z considered a parameter.

Take $z = z_0$. Then,

$$f'_{z_0}(w) = \frac{\partial P}{\partial w}(z_0, w),$$

so that $f'_{z_0}(w_0) = \frac{\partial P}{\partial w}(z_0, w_0) \neq 0$.

$\Longrightarrow f_{z_0}$ is 1:1 in an open neighb. of w_0.

Let D_2 be a disc centered at w_0 such that ∂D_2 is contained in that neighb. This means <u>in</u> particular that w_0 is the only zero of f_{z_0} in $\overline{D_2}$. Also, since ∂D_2 is cpt and $|f_{z_0}|$ is continuous on ∂D_2, it reaches a minimal value on $\partial D_2 \Longrightarrow \exists\, \delta > 0$ with $|f_{z_0}| > 2\delta$ on ∂D_2. By continuity of $P(z,w)$ in z, this means that $|f_z| > \delta$ on ∂D_2 for z close enough to z_0, say in a disc D_1 centered at z_0.

Note that by the claim, we then have: (E)

$$N(z) := \frac{1}{2\pi i} \int_{\partial D_2} \frac{f_z'(w)}{f_z(w)}\, dw$$

$$= \begin{pmatrix} \text{\# of zeros of} \\ f_z \text{ in } D_2 \end{pmatrix},$$

which is continuous in D_1. Since $N(z)$ takes values in $\mathbb{Z}^{\geq 0}$ and $N(z_0) = 1$, by continuity, we must have: $N(z) = 1$, $\forall z \in D_1$.

$$\Longrightarrow \forall z \in D_1, \text{ let } \phi(z) := \begin{pmatrix} \text{unique zero} \\ \text{of } f_z \text{ in } D_2 \end{pmatrix}.$$

Then, ϕ is a cpx fct defined on D_2. Moreover, by the claim,

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial D_2} \frac{w f_z'(w)}{f_z(w)}\, dw$$

$$= \frac{1}{2\pi i} \int_{\partial D_2} \frac{w}{P(z,w)} \frac{\partial P}{\partial w}(z,w)\, dw,$$

which is clearly holomorphic in z. ∎