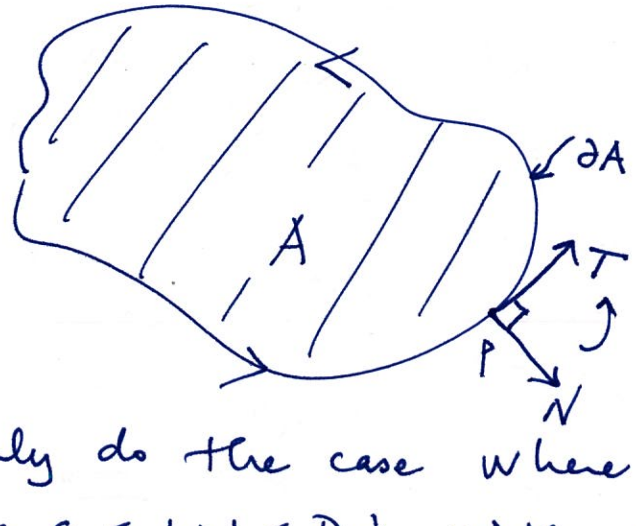
THM: (Stokes' Theorem).

Let $U \subset \mathbb{C}$ be open and $A \subset U$ be cpt with smooth boundary ∂A. Then, $\forall\, \omega \in \mathcal{E}^{(1)}(U)$,

$$\iint_A d\omega = \int_{\partial A} \omega,$$

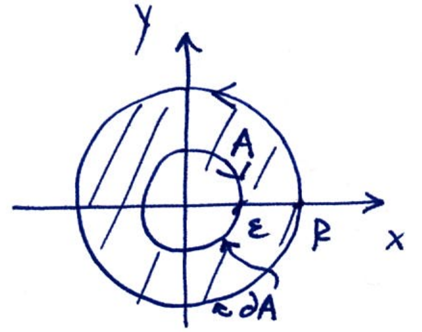
where ∂A is oriented as follows: $\forall\, p \in \partial A$, let N be the unit outward pointing normal vector (i.e. pointing out of A) and T be the unit tangent vector to ∂A pointing in the direction of the orientation. Then $\{N, T\}$ is a positively oriented basis of $\mathbb{R}^2$ (i.e. T is obtained from N by rotating by $90°$ counterclockwise).



Pf: We will only do the case where $\omega = g\, dy$. and $A = \{ z \in \mathbb{C} : \varepsilon \le |z| \le R \}$ with $0 < \varepsilon < R$.

Then:

$\partial A =$ 



Now, $d\omega = \frac{\partial g}{\partial x}\, dx \wedge dy$. Change to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Then:

$$dx \wedge dy = r\, dr \wedge d\theta$$

and

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

But, $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$ since $r = \sqrt{x^2+y^2}$.

Moreover, $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$ since

$$\begin{aligned}1 = \frac{\partial x}{\partial x} &= \frac{\partial}{\partial x}(r\cos\theta)\\ &= \frac{\partial r}{\partial x}\cdot\cos\theta - r\sin\theta\frac{\partial \theta}{\partial x}\\ &= \cos^2\theta - r\sin\theta\frac{\partial \theta}{\partial x}.\end{aligned}$$

$$\Rightarrow \sin^2\theta = -r\sin\theta\frac{\partial\theta}{\partial x}$$

so $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$.

Hence, in polar coordinates,

$$d\omega = \left[\cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right] dr \wedge d\theta,$$

and

$$A = \{(r,\theta) \mid \varepsilon \le r \le R,\ 0 \le \theta \le 2\pi\},$$

so that

$$\begin{aligned}\iint_A d\omega &= \int_0^{2\pi}\int_\varepsilon^R \left[\cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right] r\,dr\,d\theta.\\ &= \int_0^{2\pi}\int_\varepsilon^R \left[r\cos\theta\frac{\partial g}{\partial r} - \sin\theta\frac{\partial g}{\partial\theta}\right] dr\,d\theta\end{aligned}$$

Note that

$$\left[\cos\theta \frac{\partial}{\partial r}(rg) - \frac{\partial}{\partial\theta}(\sin\theta\, g)\right] =$$

$$\left(\cancel{\cos\theta \cdot g} + \cos\theta \cdot r \cdot \frac{\partial g}{\partial r}\right) - \left(\cancel{\cos\theta \cdot g} + \sin\theta \frac{\partial g}{\partial\theta}\right)$$

$$= r\cos\theta \frac{\partial g}{\partial r} - \sin\theta \frac{\partial g}{\partial\theta}.$$

Thus,

$$\iint_A dw = \int_0^{2\pi}\int_\varepsilon^R \left[\cos\theta \cdot \frac{\partial}{\partial r}(rg) - \frac{\partial}{\partial\theta}(\sin\theta\, g)\right] dr\, d\theta$$

$$= \int_0^{2\pi}\left(\int_\varepsilon^R \cos\theta \cdot \frac{\partial}{\partial r}(rg)\, dr\right) d\theta$$

$$- \int_\varepsilon^R \left(\int_0^{2\pi} \frac{\partial}{\partial\theta}(\sin\theta\, g)\, d\theta\right) dr$$

$$= \int_0^{2\pi}\left[\cos\theta \cdot rg \Big|_{r=\varepsilon}^{r=R}\right] d\theta$$

$$- \underbrace{\cancel{\int_\varepsilon^R \left[\sin\theta\, g \Big|_{\theta=0}^{\theta=2\pi}\right] dr}}_{0}$$

$$= \int_0^{2\pi} g(R,\theta)\, R\cos\theta\, d\theta - \int_0^{2\pi} g(\varepsilon,\theta)\, \varepsilon\cos\theta\, d\theta.$$

However, we can parametise the circle $|z| = R$ counterclockwise as:

$$(x,y) = (R\cos\theta, R\sin\theta), \quad 0 \le \theta \le 2\pi,$$

in which case

$$\oint_{|z|=R} \omega = \oint_{|z|=R} g\,dy = \int_0^{2\pi} g(R\cos\theta, R\sin\theta)\cdot d(R\sin\theta)$$

$$= \int_0^{2\pi} g(R,\theta)\cdot R\cos\theta\, d\theta.$$

Similarly, $(x,y) = (\varepsilon\cos\theta, \varepsilon\sin\theta)$, $0 \le \theta \le 2\pi$, gives a counterclockwise parametrisation of $|z| = \varepsilon$ and

$$\oint_{|z|=\varepsilon} \omega = \int_0^{2\pi} g(\varepsilon,\theta)\,\varepsilon\cos\theta\, d\theta.$$

However, ∂A is oriented so that $|z| = R$ is oriented counterclockwise <u>and</u> $|z| = \varepsilon$ is oriented clockwise.

Thus,

$$\iint_A d\omega = \int_0^{2\pi} g(R,\theta) R\cos\theta\, d\theta - \int_0^{2\pi} g(\varepsilon,\theta)\varepsilon\cos\theta\, d\theta$$

$$= \int_{\partial A} \omega.$$

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