Ex: 4) $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

$$z \longmapsto \begin{cases} \dfrac{az+b}{cz+d}, & z \neq \infty \\ \\ \dfrac{a}{c}, & z = \infty \end{cases}$$

is an automorphism whenever $ad - bc \neq 0$ (with $a, c \neq 0$).
[It is the extension of the Möbius transformation $z \longmapsto \dfrac{az+b}{cz+d}$ to $\mathbb{P}^1$.]

Pf: Note that
$$-d/c \longmapsto \infty$$
$$-b/a \longmapsto 0$$
$$\infty \longmapsto a/c.$$

Let us cover the first copy of $\mathbb{P}^1$ by the following charts:

$$\varphi : U := (\mathbb{C} - \{-d/c\}) \cup \{\infty\} \longrightarrow \mathbb{C}$$
$$z \longmapsto \begin{cases} 1/(z + d/c), & z \neq \infty \\ 0, & z = \infty \end{cases}$$

and

$$\varphi' : U' := (\mathbb{C} - \{-b/a\}) \cup \{\infty\} \longrightarrow \mathbb{C}$$
$$z \longmapsto \begin{cases} 1/(z + b/a), & z \neq \infty \\ 0, & z = \infty. \end{cases}$$

Note that $d/c \neq b/a$ since $ad - bc \neq 0$, so that $\mathbb{P}^1 = U \cup U'$.

We then cover the second copy of $\mathbb{P}^1$ by the standard charts:

$$\psi : V = \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto z$$

and

$$\psi' : V' = \mathbb{C}^* \cup \{\infty\} \longrightarrow \mathbb{C}$$
$$z \longmapsto \begin{cases} 1/z, & z \neq \infty \\ 0, & z = \infty. \end{cases}$$

THEN, on $\mathcal{U}$:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathbb{C} \arrow[d, "\psi"] \\ \mathbb{C} \arrow[r, "\psi \circ f \circ \varphi^{-1}"'] & \mathbb{C} \end{tikzcd}$$

with $\psi \circ f \circ \varphi^{-1}(w) = f\left(\frac{1}{w} - \frac{d}{c}\right) = a - \frac{1}{c}(ad-bc)\, w$, which is hol. in w.

On $\mathcal{U}'$:

$$\begin{tikzcd} \mathcal{U}' \arrow[r, "f"] \arrow[d, "\varphi'"'] & \mathbb{C}^* \cup \{\infty\} \arrow[d, "\psi'"] \\ \mathbb{C} \arrow[r, "\psi' \circ f \circ (\varphi')^{-1}"'] & \mathbb{C} \end{tikzcd}$$

with $\psi' \circ f \circ (\varphi')^{-1}(w) = \psi'\left(f\left(\frac{1}{w} - \frac{b}{a}\right)\right) = \frac{c}{a} + \frac{1}{a^2}(ad-bc)u$, which is also hol. in w.

$$\Longrightarrow f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \text{ is hol.}$$

Moreover, the Möbius transformation has an inverse given by

$$f^{-1}(z) = \begin{cases} \dfrac{dz - b}{-cz + a}, & z \neq \infty \\ -d/c, & z = \infty, \end{cases}$$

which is also a hol. mapping from $\mathbb{P}^1$ to $\mathbb{P}^1$.

$\Rightarrow$ f is an automorphism.

$\boxtimes$