THM: ($\bar\partial$-Poincaré OR Dolbeault lemma)

Suppose that $g \in \mathcal{E}(\bar\Delta)$. Then,

$$f(z) := \frac{1}{2\pi i} \iint_\Delta \frac{g(w)}{(w-z)} \, dw \wedge d\bar{w}$$

is defined on Δ. Moreover, $f \in \mathcal{E}(U)$ and $\frac{\partial f}{\partial \bar z} = g$.

NOTE: Here, Δ is any disc in $\mathbb{C}$. In fact, we can take $\Delta = \mathbb{C}$.

Pf: Let $z_0 \in \Delta$. and choose ε such that the disc

$$\Delta(z_0, 2\varepsilon) := \{ z \in \mathbb{C} \mid |z - z_0| < 2\varepsilon \}$$

is included in Δ. Also consider the disc

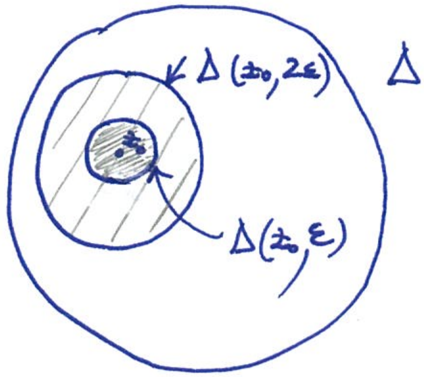
$$\Delta(z_0, \varepsilon) := \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}.$$

By using bump functions, we can write

$$g(z) = g_1(z) + g_2(z),$$

where g_1 vanishes outside $\Delta(z_0, 2\varepsilon)$ and g_2 vanishes inside $\Delta(z_0, \varepsilon)$.

We define



$$f_2(z) := \frac{1}{2\pi i} \iint_\Delta \frac{g_2(w)}{(w-z)} \, dw \wedge d\bar{w}$$

for all $z \in \Delta(z_0, \varepsilon)$. Now that since g_2 vanishes on $\Delta(z_0, \varepsilon) \subset \Delta$, this integral is well-defined for all $z \in \Delta(z_0, \varepsilon)$. Moreover,

$$\frac{\partial f_2}{\partial z}(z) = \frac{1}{2\pi i} \iint_\Delta \frac{\partial}{\partial z}\left(\frac{g_2(w)}{w-z} \right) dw \wedge d\bar{w}$$

$$= \frac{1}{2\pi i} \iint_\Delta \frac{g_2(w)}{(w-z)^2} \, dw \wedge d\bar{w}$$

and

$$\frac{\partial f_2}{\partial \bar{z}}(z) = \frac{1}{2\pi i} \iint_\Delta \cancel{\frac{\partial}{\partial \bar{z}}\left(\frac{g_2(w)}{w-z} \right)}^{0} dw \wedge d\bar{w} = 0$$

for all $z \in \Delta(z_0, \varepsilon)$. Note that the integral giving $\frac{\partial f_2}{\partial z}$ is always defined since g_2 vanishes on $\Delta(z_0, \varepsilon)$ and $z \in \Delta(z_0, \varepsilon)$. One check similarly that all higher order partial derivatives of f_2 exist, implying that $f_2 \in \mathcal{E}(\Delta(z_0, \varepsilon))$. We have thus constructed $f_2 \in \mathcal{E}(\Delta(z_0, \varepsilon))$ with $\frac{\partial f_2}{\partial \bar{z}} = 0 = g_2$ on $\Delta(z_0, \varepsilon)$, where

$$f_2(z) = \frac{1}{2\pi i} \iint_\Delta \frac{g_2(w)}{(w-z)} \, dw \wedge d\bar{w}.$$

Let us now define

$$f_1(z) := \frac{1}{2\pi i} \iint_\Delta \frac{g_1(w)}{(w-z)} \, dw \wedge d\bar{w},$$

for all $z \in \Delta(z_0, \varepsilon)$. We want to verify that this integral is well-defined for all $z \in \Delta(z_0, \varepsilon)$. We first note that since g_1 vanishes outside $\Delta(z_0, 2\varepsilon)$,

$$f_1(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g_1(w)}{(w-z)} \, dw \wedge d\bar{w}.$$

Setting $w = z + re^{i\theta}$ with $r \in \mathbb{R}^{>0}$ and $\theta \in [0, \pi]$, we have: $g_1(w) = g_1(z + re^{i\theta})$, $(w - z) = re^{i\theta}$ and

$$\begin{aligned} dw \wedge d\bar{w} &= d(z + re^{i\theta}) \wedge d(\bar{z} + re^{-i\theta}) \\ &= [e^{i\theta} dr + ire^{i\theta} d\theta] \wedge [e^{-i\theta} dr - ire^{-i\theta} d\theta] \\ &= -2ir \, dr \wedge d\theta. \end{aligned}$$

Thus, $f_1(z) = -\frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} g_1(z + re^{i\theta}) e^{-i\theta} \, dr \wedge d\theta$, will is well-defined for all $z \in \Delta(z_0, \varepsilon)$ since $g_1 \in \mathcal{E}(\bar{\Delta})$ and g_1 vanishes outside $\Delta(z_0, 2\varepsilon) \subset \Delta$.

Moreover, we checks as for f_2 that $f_1 \in \mathcal{E}(\Delta(z_0, \varepsilon))$ In particular,

$$\frac{\partial f_1}{\partial \bar{z}}(z) = -\frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \frac{\partial}{\partial \bar{z}} \left(g_1(z + re^{i\theta}) \right) e^{-i\theta} \, dr \wedge d\theta.$$

BUT, since $w = z + re^{i\theta}$,

$$\frac{\partial}{\partial \bar{z}} \left(g_1(z + re^{i\theta}) \right) = \frac{\partial g_1}{\partial w} \cdot \overset{0}{\cancel{\frac{\partial w}{\partial \bar{z}}}} + \frac{\partial g_1}{\partial \bar{w}} \cdot \underbrace{\frac{\partial \bar{w}}{\partial \bar{z}}}_{=1} = \frac{\partial g_1}{\partial \bar{w}}(w)$$

Thus,

$$\frac{\partial f_1}{\partial \bar{z}}(z) = \frac{-1}{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\partial g_1}{\partial \bar{w}} \cdot \underbrace{e^{-i\theta} dr \wedge d\theta}_{\substack{\| \\ -\frac{1}{2i} \cdot \frac{dw \wedge d\bar{w}}{(w-z)}}}$$

$$= \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\partial g_1}{\partial \bar{w}} \cdot \frac{dw \wedge d\bar{w}}{(w-z)}.$$

$$= g_1(z) - \oint_{\partial\Delta} \frac{g_1(w)}{(w-z)}\, dw$$

by the Cauchy Int. formula.

BUT, g_1 is zero on $\partial\Delta$ since it vanishes outside $\Delta(z_0, 2\varepsilon) \subset \Delta$. Thus, $\oint_{\partial\Delta} \frac{g_1(w)}{(w-z)}\, dw = 0$ and

$$\frac{\partial f_1}{\partial \bar{z}} = g_1(z), \quad \forall z \in \Delta(z_0, \varepsilon).$$

Hence, $\forall z \in \Delta(z_0, \varepsilon)$, $f(z) = f_1(z) + f_2(z)$ is well-defined and $f \in \mathcal{E}(\Delta(z_0, \varepsilon))$. Moreover,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f_1}{\partial \bar{z}} + \cancelto{0}{\frac{\partial f_2}{\partial \bar{z}}} = g_1(z) = g_1(z) + \cancelto{0}{g_2(z)} = g(z)$$

for all $z \in \Delta(z_0, \varepsilon)$. Since the choice of z_0 was arbitrary and $\Delta(z_0, \varepsilon) \subset \Delta$, the result must hold on all Δ. $\blacksquare$