PMATH 800: Assignment 4

Due: Friday, 4 April, 2014

1. Let X be a Riemann surface. Consider the skyscraper sheaf $\mathbb{C}_p$ supported at $p \in X$ (see Assignment 2). Prove that $H^0(X, \mathbb{C}_p) = \mathbb{C}$ and $H^1(X, \mathbb{C}_p) = 0$.

2. Let X be a Riemann surface and $U \subset X$ be an open subset. A differentiable function $f \in \mathcal{E}(U)$ is called *harmonic* if and only if $\partial\bar{\partial}f = 0$. Let $\mathcal{H}$ be the sheaf of harmonic functions on X, where $\mathcal{H}(U)$ is the set of all harmonic functions on U and the restriction maps are the usual restrictions of functions. Show that the sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{E} \xrightarrow{\partial\bar{\partial}} \mathcal{E}^{(2)} \longrightarrow 0$$

is exact.

3. *Complex line bundles.* Let X be a Riemann surface. A *complex line bundle* on X is a topological space L together with a surjective continuous map $\pi : L \to X$ that satisfy the following properties:

   - For every $x \in X$, the set $L_x = \pi^{-1}(x) \subset L$ (called the *fibre* of L over p) is endowed with the structure of a k-dimensional $\mathbb{C}$-vector space.
   - For every $x \in X$ admits an open neighbourhood U and a homeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}$ (called a *local trivialisation* of L over U) such that $\varphi_U|_{L_y} : L_y \to \{y\} \times \mathbb{C}^k$ is a linear isomorphism for all $y \in U$.

   If the local trivialisations can be chosen so that

$$\begin{array}{rcl} \varphi_U \circ \varphi_V^{-1} : (U \cap V) \times \mathbb{C} & \longrightarrow & (U \cap V) \times \mathbb{C} \\ (z, v) & \longmapsto & (z, g_{UV}(z)v) \end{array}$$

   with $g_{UV} : U \cap V \to \mathbb{C}^*$ is holomorphic map, then L is said to be a *holomorphic* line bundle. The map g_{UV} is called the *transition functions* of L between the local trivialisations φ_U and φ_V.

   Note: Line bundles are vector bundles of rank one. For more facts on vector bundles, see chapter 5 of John M. Lee's book "Introduction to Smooth Manifolds", which is available free online through the UWaterloo library.

   (i) Let $K_X := \sqcup_{x \in X} T_x^{1,0}$, where $T_x^{1,0}$ is the space of cotangent (1,0)-forms at x on X. Show that K_X is a holomorphic line bundle on X, called the *canonical line bundle of* X.

   (ii) Let L be a holomorphic line bundle on X. Moreover, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X with respect to which L admits local trivialisations $\varphi_i : \pi^{-1}(U_i) \to U \times \mathbb{C}$ with corresponding transition functions $g_{ij} := g_{U_i U_j}$. Show that $\{g_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$, so that $[\{g_{ij}\}] \in H^1(X, \mathcal{O}^*)$. Conversely, let $\{h_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ and consider

$$\widetilde{L} := (\sqcup_{i \in I} U_i \times \mathbb{C}) / \sim,$$

   where two elements $(u_i, t_i) \in U_i \times \mathbb{C}$ and $(u_j, t_j) \in U_j \times \mathbb{C}$ are equivalent if and only if $u_i = u_j = p \in U_i \cap U_j$ and $t_i = h_{ij}(p)t_j$. Show that $\sim$ is indeed an equivalence relation and that $\widetilde{L}$ is a holomorphic line bundle on X.

   Note: For every open set $U \subset X$, the set $\mathcal{O}^*(U)$ of nowhere vanishing holomorphic functions on U is an abelian group under function multiplication. Hence, since the abelian structure on $\mathcal{O}^*(U)$ is multiplicative, the *cocycle relation* is $g_{ij} \cdot g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$ for all $i, j, k \in I$.

(iii) Let L and $\widetilde{L}$ be two line bundles on X. Suppose that $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X with respect to which both L and $\widetilde{L}$ admit local trivialisations with corresponding transition functions g_{ij} and $\widetilde{g_{ij}}$, respectively. Then, L and $\widetilde{L}$ are said to be *isomorphic*, denoted $L \cong \widetilde{L}$, if and only if there exist holomorphic maps $g_i : U_i \to \mathbb{C}^*$ such that

$$\widetilde{g_{ij}}(p) = g_i(p)g_{ij}(p)g_j(p)^{-1}$$

for all $p \in U_i \cap U_j$. Show that this definition is independent of the open cover $\mathcal{U}$. Moreover, show that $H^1(X, \mathcal{O}^*)$ parametrises holomorphic line bundles on X up to isomorphism. The cohomology group $H^1(X, \mathcal{O}^*)$ is called the *Picard group of* X and is denoted $Pic(X)$.

(iv) In this part of the question, we prove that $Pic(\mathbb{P}^1) = \mathbb{Z}$, implying that holomorphic line bundles on $\mathbb{P}^1$ are classified by the integers, up to isomorphism. We will need the following:

Fact: Every holomorphic line bundle L on a non-compact Riemann surface X admits a *global* trivialisation $\varphi : L \to X \times \mathbb{C}$ (that is, a local trivialisation with $U = X$). (For a proof of this, see Forster, Theorem 30.3, p. 229.)

- Let $\mathcal{U} = \{U_1, U_2\}$ be the standard open cover of $\mathbb{P}^1$, where $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}^* \cup \{\infty\}$. Use the above fact to show that $\mathcal{U}$ is a Leray cover of $\mathcal{O}^*$ on $\mathbb{P}^1$. Since $\mathcal{U}$ only consists of two open sets, this means in particular that every element in $Z^1(\mathbb{P}^1, \mathcal{O}^*)$ is completely determined by a section $g_{12} \in \mathcal{O}^*(U_1 \cap U_2)$.
- Consider the map

$$\begin{aligned} \deg : Pic(\mathbb{P}^1) &\longrightarrow \mathbb{Z} \\ [g_{12}] &\longmapsto \frac{1}{2\pi i}\int_C \frac{g_{12}'(z)}{g_{12}(z)}dz, \end{aligned}$$

  where C is the unit circle in $U_1 \cap U_2 = \mathbb{C}^*$ and g_{12}' is the usual derivative with respect to z. Show that this map is a well-defined group isomorphism (you will in particular have to check that $\deg([g_{12}])$ is independent of the representative g_{12}, and also an integer!).

Let L be a holomorphic line bundle on $\mathbb{P}^1$. If L corresponds, up to isomorphism, to the element $[g_{12}]$, we define the *degree of* L to be $\deg(L) := \deg([g_{12}])$. Since any holomorphic line bundle on $\mathbb{P}^1$ is determined up to isomorphism by its degree, we denote by $\mathcal{O}(n)$ any holomorphic line bundle on $\mathbb{P}^1$ of degree n.

(v) Prove that $\deg(K_{\mathbb{P}^1}) = -2$, so that $K_{\mathbb{P}^1} \cong \mathcal{O}(-2)$.