PMATH 800: Assignment 3

Due: Friday, 21 March, 2014

1. Prove that the holomorphic 1-form
$$\frac{dz}{1+z^2},$$
which is defined on $\mathbb{C}\backslash\{\pm i\}$, can be extended to a holomorphic 1-form ω on $\mathbb{P}^1\backslash\{\pm i\}$. Let
$$p := \tan : \mathbb{C} \to \mathbb{P}^1\backslash\{\pm i\}$$
(see problem 3 on Assignment 2) and find $p^*\omega$.

2. *Poincaré lemma.* The goal of this problem is to prove the Poincaré lemma which states that $H^1_{dR}(\mathbb{C}) = H^2_{dR}(\mathbb{C}) = 0$.

   (a) Let $\omega = fdx + gdy$ be a d-closed differentiable 1-form on $\mathbb{C}$. Verify that $\omega = dh$, where
$$h(x, y) := \int_0^1 [xf(tx, ty) + yg(tx, ty)]dt$$
   is a differentiable function on $\mathbb{C}$.

   (b) Let $\omega = fdx \wedge dy$ be a differentiable 2-form on $\mathbb{C}$. Verify that $\omega = d\alpha$, where
$$\alpha := \left(-\int_0^1 tyf(tx, ty)dt\right) dx + \left(\int_0^1 txf(tx, ty)dt\right) dy$$
   is a differentiable 1-form on $\mathbb{C}$.

   (c) Use (i) and (ii) to prove the Poincaré lemma.

3. Let $T = \mathbb{C}/\Gamma$ for some lattice $\Gamma \subset \mathbb{C}$. In this problem, we prove that $H^1_{dR}(T) = \mathbb{C}^2$ and $H^{1,0}_{\bar\partial}(T) = \mathbb{C}$.

   (a) Show that the differentiable 1-forms dz and $d\bar z$ on $\mathbb{C}$ descend to differentiable 1-forms on T.

   (b) Suppose that Γ is spanned by ω_1 and ω_2, and let $\pi : \mathbb{C} \to T$ be the natural projection. Let $\hat\gamma_i$ be the side of the fundamental parallelogram joining 0 and ω_i, $i = 1, 2$. Then, $\gamma_1 := \pi \circ \hat\gamma_1$ and $\gamma_2 = \pi \circ \hat\gamma_2$ are closed curves on T (that generate the fundamental group of T). Set
$$\begin{array}{rcl} P : H^1_{dR}(T) & \longrightarrow & \mathbb{C}^2 \\ [\alpha] & \longmapsto & (\int_{\gamma_1} \alpha, \int_{\gamma_2} \alpha). \end{array}$$
   Show that P is a well-defined linear isomorphism, proving that $H^1_{dR}(T) = \mathbb{C}^2$.

   (c) Use (ii) to show that $H^{1,0}_{\bar\partial}(T) = \mathbb{C}$.

4. *Dolbeault lemma.* Prove that $H^{0,1}_{\bar\partial}(\mathbb{C}) = H^{1,1}_{\bar\partial}(\mathbb{C}) = 0$.