PMATH 800: Assignment 2

Due: Wednesday, 26 February, 2014

1. *Open Mapping Theorem.* The Open Mapping Theorem on $\mathbb{C}$ states that if $D \subset \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is a non-constant holomorphic function, then f is open. Show that the theorem extends to Riemann surfaces: If $f : X \to Y$ is a non-constant holomorphic mapping between Riemann surfaces X and Y, then f is open.

2. *Elliptic functions.* Let $\Gamma = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ be a lattice in $\mathbb{C}$. An *elliptic function (relative to the lattice* Γ*)* is a doubly periodic meromorphic function with respect to Γ.

   (a) Let $\mathbb{C}(\Gamma)$ be the set of all elliptic functions. Show that $\mathbb{C}(\Gamma)$ is a field.

   (b) The *Weierstrass $\mathcal{P}$-function* with respect to Γ is defined by

   $$\mathcal{P}_\Gamma(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \backslash 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

   Show that $\mathcal{P}_\Gamma$ is an elliptic function relative to Γ which has poles at the points of Γ. [*Hint:* First consider the derivative $\mathcal{P}'_\Gamma(z) = -2\sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3}$.]

   (c) Let $f \in \mathbb{C}(\Gamma)$ be an elliptic function relative to Γ that has its poles at the points of Γ and which has the following Laurent series expansion about the origin

   $$f(z) = \sum_{k=-2}^{\infty} c_k z^k,$$

   with $c_{-2} = 1$ and $c_{-1} = c_0 = 0$. Prove that $f = \mathcal{P}_\Gamma$.
   Note: One can show that

   $$\mathbb{C}(\Gamma) = \mathbb{C}(\mathcal{P}_\Gamma(z), \mathcal{P}'_\Gamma(z)).$$

   In other words, every elliptic function is a rational combination of $\mathcal{P}_\Gamma$ and $\mathcal{P}'_\Gamma$. See for example "Elliptic functions" by Serge Lang (on reserve at the DC Library) for details.

   (d) The *order* of an elliptic function is its number of poles (counted with multiplicity) in any fundamental parallelogram $\{\omega + \alpha\omega_1 + \beta\omega_2 : 0 \le \alpha, \beta < 1\}$, $\omega \in \Gamma$. Show that a non-constant elliptic function has order at least 2. This implies in particular that, if an elliptic function has a single pole, then its pole must have multiplicity at least 2.

3. (a) Show that

   $$\tan : \mathbb{C} \to \mathbb{P}^1$$

   is a local homeomorphism.

   (b) Show that $\tan(\mathbb{C}) = \mathbb{P}^1 \backslash \{\pm i\}$ and

   $$\tan : \mathbb{C} \to \mathbb{P}^1 \backslash \{\pm i\}$$

   is a covering map.

   (c) Let $X = \mathbb{C} \backslash \{it : t \in \mathbb{R}, |t| \ge 1\}$. Show that for every $k \in \mathbb{Z}$ there exists a unique holomorphic function $\arctan_k : X \to \mathbb{C}$ with

   $$\tan \circ \arctan_k = \mathrm{id}_X$$

   and

   $$\arctan_k(0) = k\pi$$

   (the k-th branch of arctan).

4. Let X be a compact Riemann surfaces.

   (a) Let f be non-constant meromorphic function on X. Show that f has the same number of zeroes and poles, counting multiplicities.

   (b) Suppose that there exists a meromorphic function f on X that has a single pole, and that this pole has multiplicity one. Prove that X is isomorphic to $\mathbb{P}^1$. Moreover, prove that if X is isomorphic to $\mathbb{P}^1$, then X admits a meromorphic function f that has a single pole, and that this pole has multiplicity one.

   (c) Use (b) to show that $\mathbb{C}/\Gamma$ cannot be isomorphic to $\mathbb{P}^1$ for any lattice $\Gamma \subset \mathbb{C}$.

5. *Sheafification.* Let $\mathcal{F}$ be a presheaf on the topological space X. Let $|\mathcal{F}| = \coprod_{x\in X} \mathcal{F}_x$, denote

$$\begin{array}{rcl} p : |\mathcal{F}| & \to & X \\ \mathcal{F}_x \ni \varphi & \mapsto & x \end{array}$$

   the natural projection map. Let $U \subset X$ be a subset. A map $s : U \mapsto |\mathcal{F}|$ such that $p \circ s = \mathrm{id}_U$ is called a *section of p over U*.

   (a) Let $s : U \mapsto |\mathcal{F}|$ be a section of p over U. Show that s is continuous if and only if it satisfies the condition: for every $x \in U$, there exist an open neighbourhood V of x and a section $t \in \mathcal{F}(V)$ such that $s(y) = \rho_y(t)$ for all $y \in V$.

   (b) We associate the following presheaf $\mathcal{F}^+$ to $\mathcal{F}$: for any open set $U \subset X$, let $\mathcal{F}^+(U)$ be the set of all continuous section $s : U \to |\mathcal{F}|$ of p over U; moreover, if $V \subset U$ is an open subset, let $\rho_V^U(s) = s|_V$ be usual restriction of functions. Show that $\mathcal{F}^+$ is a sheaf of abelian group on X, called the *sheafification of $\mathcal{F}$*.

   (c) Prove that there is a natural isomorphism of the stalks $\mathcal{F}_x \simeq \mathcal{F}_x^+$ for all $x \in X$.

   (d) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two sheaves of abelian groups on X. A *sheaf homomorphism* $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ is a family of group homomorphisms $\alpha_U : \mathcal{G}_1(U) \to \mathcal{G}_2(U)$, for every $U \subset X$ open, which are compatible with the restriction homomorphisms, that is, if $V \subset U$ is open and $(\rho_1)_V^U$, $(\rho_2)_V^U$ are the restriction homomorphisms of $\mathcal{G}_1$, $\mathcal{G}_2$, respectively, then $(\rho_2)_V^U \circ \alpha_U = \alpha_V \circ (\rho_1)_V^U$. If all the α_U are isomorphisms, then α is called an *isomorphism*, and $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *isomorphic*, which is denoted $\mathcal{G}_1 \simeq \mathcal{G}_2$.

   Show that if $\mathcal{F}$ is a sheaf, then $\mathcal{F} \simeq \mathcal{F}^+$.

   Hint: Let $U \subset X$ be an open. For any $f \in \mathcal{F}$, let $\hat{f} : U \to |\mathcal{F}|, x \mapsto \rho_x(f)$. Then, $\hat{f} \in \mathcal{F}^+(U)$. Consider the map $\alpha_U : \mathcal{F}(U) \to \mathcal{F}^+(U), f \mapsto \hat{f}$.

6. Let X be a Riemann surface.

   (a) Let $p \in X$. Consider the following collection $\mathbb{C}_p$ of abelian groups

$$\mathbb{C}_p(U) := \begin{cases} \mathbb{C}, & p \in U, \\ 0, & p \notin U, \end{cases}$$

   for any open set $U \subset X$, and restriction homomorphisms

$$\rho_V^U(f) := \begin{cases} f, & p \in V, \\ 0, & p \notin V, \end{cases}$$

   for all $f \in \mathbb{C}_p(U)$ and open sets $V \subset U$. Show that $\mathbb{C}_p$ is a sheaf, called the *skyscraper sheaf supported at p*, whose stalks are given by $(\mathbb{C}_p)_x = \mathbb{C}$ if $x = p$, and $(\mathbb{C}_p)_x = 0$ if $x \neq p$.

   (b) For any open set $U \subset X$, define

$$\mathcal{F}(U) := \begin{cases} \mathbb{C}, & U \neq \emptyset, \\ 0, & U = \emptyset, \end{cases}$$

and restriction homomorphisms

$$\rho_V^U(f) := \begin{cases} f, & V \neq \emptyset, \\ 0, & V = \emptyset, \end{cases}$$

for all $f \in \mathcal{F}(U)$ and open sets $V \subset U$. Show that $(\mathcal{F}, \rho)$ is a presheaf, but not a sheaf. Recall that a function $f : X \to \mathbb{C}$ is called *locally constant* if, for every $x \in X$, there exists and open neighbourhood W of x such that $f|W$ is constant. Prove that the sheafification $\mathcal{F}^+$ of $\mathcal{F}$ is the sheaf $\mathbb{C}$ of locally constant functions on X (with values in $\mathbb{C}$), where $\mathbb{C}(U)$ is the set of locally constant functions on the open set $U \subset X$, and the restriction homomorphisms are the usual restriction of functions.

(c) For $U \subset X$ open, let

$$\mathcal{F} := \mathcal{O}^*(U)/\exp \mathcal{O}(U).$$

Show that $\mathcal{F}$ with the usual restriction maps is a presheaf, but not a sheaf.