PMATH 800: Assignment 1

Due: Friday, 31 January, 2014

1. (a) *One point compactification of* $\mathbb{R}^n$. For $n \geq 1$, let ∞ be a symbol not belonging to $\mathbb{R}^n$. Introduce the following topology on the set $X := \mathbb{R}^n \cup \{\infty\}$. A set $U \subset X$ is open, by definition, if one of the two following conditions is satisfies:

   i. $\infty \notin U$ and U is open in $\mathbb{R}^n$ with respect to the usual topology on $\mathbb{R}^n$.

   ii. $\infty \in U$ and $K := X \backslash U$ is compact in $\mathbb{R}^n$ with respect to the usual topology on $\mathbb{R}^n$.

   Show that X is a compact Hausdorff topological space.

   (b) *Stereographic projection.* Consider the unit n-sphere

   $$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

   and the stereographic projection

   $$\sigma : S^n \to \mathbb{R}^n \cup \{\infty\}$$

   given by

   $$\sigma(x_1, \ldots, x_{n+1}) := \begin{cases} \dfrac{1}{1 - x_{n+1}}(x_1, \ldots, x_n), & \text{if } x_{n+1} \neq 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases}$$

   Show that σ is a homeomorphism of S^n onto X; this proves in particular that the Riemann sphere $\mathbb{P}^1$ is homeomorphic to S^2 (after identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the natural way).

2. *The complex projective line.* The *complex projective line* is defined as the quotient

   $$\mathbb{CP}^1 := (\mathbb{C}^2 - \{0\}) / \sim,$$

   where $(z_1, z_2) \sim (\alpha z_1, \alpha z_2)$ for all $\alpha \in \mathbb{C}^*$, and corresponds to the set of all lines through the origin 0 in $\mathbb{C}^2$.

   (a) Prove that $\mathbb{CP}^1$ endowed with the quotient topology is a topological surface.

   (b) Let $U_i := \{(z_1, z_2) : z_i \neq 0\}$, $i = 1, 2$, and consider the maps

   $$\begin{array}{rcl} \varphi_1 : U_1 & \longrightarrow & \mathbb{C} \\ (z_1, z_2) & \longmapsto & z_2/z_1 =: z \end{array}$$

   and

   $$\begin{array}{rcl} \varphi_2 : U_2 & \longrightarrow & \mathbb{C} \\ (z_1, z_2) & \longmapsto & z_1/z_2 =: z'. \end{array}$$

   Show that $\mathcal{U} := \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is a holomorphic atlas on $\mathbb{CP}^1$, thus inducing a complex structure Σ on $\mathbb{CP}^1$.

   (c) Prove that $(\mathbb{CP}^1, \Sigma)$ is isomorphic to the Riemann sphere $\mathbb{P}^1$.

3. Let $f(z)$ be a continuous complex function with real part u and imaginary part v. Recall that f is holomorphic at z_0 if and only if the partials of u and v exist and satisfy the Cauchy-Riemann equations

   $$u_x = v_y, \quad u_y = -v_x$$

   in an open neighbourhood of z_0 (this is known as Looman-Menchoff's Theorem, see).

   Note that $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$, so that u and v can be considered as functions of z and $\bar{z}$. Verify that

   $$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x),$$

   and conclude that f is holomorphic at z_0 if and only if $\partial f / \partial \bar{z} = 0$ in an open neighbourhood of z_0.

4. Let X and Y be Riemann surfaces, and $f : X \to Y$ be a holomorphic map. Consider the induced *pullback* map

$$\begin{array}{rcl} f^* : \mathcal{O}(Y) & \longrightarrow & \mathcal{O}(X) \\ \varphi & \longmapsto & \varphi \circ f. \end{array}$$

Note that $f^*(\varphi) := \varphi \circ f$ is called the *pullback of* φ *under* f. Prove that f^* is a well-defined ring homomorphism, which is a monomorphism if f is non-constant.

5. Let $\Gamma = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ and $\Gamma' = \{m\omega_1' + n\omega_2' : m, n \in \mathbb{Z}\}$ be two lattices in $\mathbb{C}$.

   (a) Prove that $\mathbb{C}/\Gamma$ is homeomorphic to $S^1 \times S^1$, where the homeomorphism is given by $\lambda\omega_1 + \mu\omega_2 \mapsto (e^{2\pi i\lambda}, e^{2\pi i\mu})$ with $\lambda, \mu \in \mathbb{R}$. Note that we consider $S^1 \times S^1$ as a subset of $\mathbb{C} \times \mathbb{C}$ with the induced metric topology, given by $\{(e^{2\pi i\lambda}, e^{2\pi i\mu}) : \lambda, \mu \in \mathbb{R}\}$; moreover, since ω_1, ω_2 are $\mathbb{R}$-linearly independent, $\mathbb{C} = \{\lambda\omega_1 + \mu\omega_2 : \lambda, \mu \in \mathbb{R}\}$.

   (b) Show that $\Gamma = \Gamma'$ if and only if there exists a matrix $A \in SL(2, \mathbb{Z}) := \{A \in GL(2, \mathbb{Z}) : \det A = 1\}$ such that

   $$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ or } \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}.$$

   (c) Suppose $\alpha \in \mathbb{C}^*$ is such that $\alpha\Gamma \subset \Gamma'$. Show that the multiplication map $\mathbb{C} \to \mathbb{C}, z \mapsto \alpha z$, induces a holomorphic map

   $$\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma',$$

   which is biholomorphic if and only if $\alpha\Gamma = \Gamma'$.

   (d) Show that every torus $X = \mathbb{C}/\Gamma$ is isomorphic to a torus of the form

   $$X(\tau) := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau),$$

   where τ satisfies $Im(\tau) > 0$.

   (e) Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $Im(\tau) > 0$. Let

   $$\tau' := \frac{a\tau + b}{c\tau + d}.$$

   Show that the tori $X(\tau)$ and $X(\tau')$ are isomorphic.

6. *(Optional)* A topological surface X is said to be *smooth* if it admits an atlas

$$\mathcal{U} = \{(U_\alpha, \varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^2) : \alpha \in A\}$$

such that

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth for all $\alpha, \beta \in A$, and the atlas $\mathcal{U}$ is called a *smooth structure* on X. Furthermore, if X admits a smooth structure $\mathcal{U}$ such that $\det(Jac(\varphi_\alpha \circ \varphi_\beta^{-1})) > 0$ for all $\alpha, \beta \in A$, then X is said to be *orientable* (with *orientation* given by $\mathcal{U}$). Prove that every Riemann surface is orientable.