Chapter 1

Projective Varieties

1.1 Projective Space and Algebraic Sets

1.1.1 Definition. Consider $\mathbb{A}^{n+1} = \mathbb{A}^{n+1}(\Bbbk)$. The set of all lines in $\mathbb{A}^{n+1}$ passing through the origin $0 = (0, \dots, 0)$ is called the n-dimensional projective space and is denoted by $\mathbb{P}^n(\Bbbk)$, or simply $\mathbb{P}^n$ when $\Bbbk$ is understood.

We also have the identification

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\Bbbk^*,$$

where $(x_1, \dots, x_{n+1}) \sim (\lambda x_1, \dots, \lambda x_{n+1})$ for all $\lambda \in \Bbbk^*$, i.e. , two points in $\mathbb{A}^{n+1} \setminus \{0\}$ are equivalent if they are on the same line through the origin. An element of $\mathbb{P}^n$ is called a point. If P is a point, then any $(n+1)$-tuple $(a_1, \dots, a_{n+1})$ in the equivalence class P is called a *set of homogeneous coordinates for* P. Equivalence classes are often denoted by $P = [a_1 : \cdots : a_{n+1}]$ to distinguish from the affine coordinates. Note that $[a_1 : \cdots : a_{n+1}] = [\lambda a_1 : \cdots : \lambda a_{n+1}]$ for all $\lambda \in \Bbbk^*$.

We defined $\mathbb{P}^n$ as the collection of all one dimensional subspaces of the vector space $\mathbb{A}^{n+1}$, but $\mathbb{P}^n$ may also be thought of as $n+1$ (overlapping) copies of affine n-space. Indeed, we can express any any point $[x_1 : \cdots : x_{n_1}] \in U_i$ in terms of n affine coordinates:

$$[x_1 : \cdots : x_{n+1}] = \left[\frac{x_1}{x_i} : \cdots : 1 : \cdots : \frac{x_{n+1}}{x_i}\right].$$

Thus, $U_i \cong \mathbb{A}^n$.

1.1.2 Example. Consider $\mathbb{P}^2$, where $[x : y : z]$ are the homogeneous coordinates. Then

$$\begin{aligned} U_x &= \left\{\left[x : \frac{y}{x} : \frac{z}{x}\right] \in \mathbb{P}^2 \mid x \neq 0\right\} \\ &\cong \{[1 : u : v] \mid u, v \in \Bbbk\} \\ &= \{(u, v) \mid u, v \in \Bbbk\} \\ &= \mathbb{A}^2, \end{aligned}$$

so (u, v) are the affine coordinates on U_x. Similarly,

$$U_y = \{[x : y : z] \in \mathbb{P}^2 \mid y \neq 0\} = \left\{ \left[\frac{x}{y} : 1 : \frac{z}{y} \right] \in \mathbb{P}^2 \mid y \neq 0 \right\},$$

and

$$U_z = \{[x : y : z] \in \mathbb{P}^2 \mid z \neq 0\} = \left\{ \left[\frac{x}{z} : \frac{y}{z} : 1 \right] \in \mathbb{P}^2 \mid y \neq 0 \right\}.$$

Moreover, $\mathbb{A}^n$ can be considered as a subspace of $\mathbb{P}^n$, where the inclusion is given by identifying $\mathbb{A}^n$ with $U_{n+1} \subseteq \mathbb{P}^n$, i.e. the inclusion is the map $\varphi : \mathbb{A}^n \to \mathbb{P}^n$ given by $\varphi(u_1, \dots, u_n) = [u_1 : \cdots : u_n : 1]$. One could also introduce a 1 to any other position, but we will usually use this convention.

For each $i = 1, \dots, n+1$, $H_i = \{[x_1 : \cdots : x_{n+1}] \mid x_i = 0\} = \mathbb{P}^n \setminus U_i$ is a *hyperplane*, which can be identified with $\mathbb{P}^{n-1}$ by the correspondence

$$[x_1 : \cdots : 0 : \cdots : x_{n+1}] \longleftrightarrow [x_1 : \cdots : x_{i-1} : x_{i+1} : \cdots : x_{n+1}].$$

In particular, H_{n+1} is often denoted H_∞ and is called the *hyperplane at infinity*, and

$$\mathbb{P}^n = U_{n+1} \cup H_\infty \cong \mathbb{A}^n \cup \mathbb{P}^{n-1},$$

so $\mathbb{P}^n$ is the union of a copy of $\mathbb{A}^n$ and H_∞, which can be seen as the set of all directions in $\mathbb{A}^n$.

1.1.3 Examples.

(i) $\mathbb{P}^0 = \{\infty\}$ is a single point.

(ii) $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{P}^0 = \mathbb{A}^1 \cup \{\infty\}$, the one point compactification of $\mathbb{A}^1$.

(iii) $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$, where the copy of $\mathbb{P}^1$ here is often referred to as the *line at infinity*, and is denoted by ℓ_∞.

Remarks. In $\mathbb{P}^n$, any two lines intersect. For example, consider two distinct parallel lines in $\mathbb{A}^2$:

$$L : au + bv + c = 0$$
$$L' : au + bv + c' = 0,$$

where $c \neq c'$. If one considers $\mathbb{A}^2$ as the subset

$$\{[u : v : 1] \in \mathbb{P}^2 \mid u, v \in \mathbb{k}\} = \{[x : y : z] \in \mathbb{P}^2 \mid x, y \in \mathbb{k}, z \neq 0\} = U_z$$

of $\mathbb{P}^2$, we can see that the equations can be rewritten as follows by letting $u = x/z$ and $v = y/z$:

$$ax + by + cz = 0$$
$$ax + by + c'z = 0.$$

Subtracting the two equations, we get $z(c - c') = 0$, so $z = 0$, as we assumed that $c \neq c'$. Hence any solution is of the form $[x : y : 0]$ and lies on the line at infinity. Substituing this value of z back into either equation gives that $ax + by = 0$, so $x = bt$ and $y = -at$ for some $t \in \Bbbk$. Therefore, the solutions to the system are

$$\{(bt, -at, 0) \mid t \in \Bbbk^*\} = [-b : a : 0],$$

which is a single point in $\mathbb{P}^2$ on the line at infinity.

1.1.4 Definition. If $f \in \Bbbk[x_1, \ldots, x_{n+1}]$, then $P = [a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$ is a *zero* of f if $f(\lambda a_1, \ldots, \lambda a_{n+1}) = 0$ for every $\lambda \in \Bbbk^*$, in which case we write $f(P) = 0$. For any $S \subseteq \Bbbk[x_1, \ldots, x_{n+1}]$, let

$$\mathrm{V_p}(S) = \{P \in \mathbb{P}^n \mid f(P) = 0 \text{ for all } f \in S\}$$

be the *zero set of* S in $\mathbb{P}^n$. Moreover, if $Y \subseteq \mathbb{P}^n$ is such that $Y = \mathrm{V_p}(S)$ for some $S \subseteq \Bbbk[x_1, \ldots, x_{n+1}]$, then we say that Y is a *projective algebraic set*. Similarly, given $Y \subseteq \mathbb{P}^n$, let

$$\mathrm{I_p}(Y) = \{f \in \Bbbk[x_1, \ldots, x_{n+1}] \mid f(P) = 0 \text{ for all } P \in Y\}$$

be the *ideal of* Y.

Remark. To avoid confusion, from now on we will use the notation $\mathrm{I_a}$ and $\mathrm{V_a}$ for the ideal of a set of points in $\mathbb{A}^n$ and the zero set in $\mathbb{A}^n$ of a set of polynomials, respectively.

1.1.5 Lemma. *Let $f \in \Bbbk[x_1, \ldots, x_{n+1}]$ be such that $f = f_m + \cdots + f_d$, where each f_i is an i-form. Then, for any $P \in \mathbb{P}^n$, $f(P) = 0$ if and only if $f_i(P) = 0$ for $i = m, \ldots, d$.*

Proof: Suppose $f(P) = 0$ for $P = [a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$. Then

$$q(\lambda) = \lambda^m f_m(a_1, \ldots, a_{n+1}) + \cdots + \lambda^d f_d(a_1, \ldots, a_{n+1}) = 0$$

for all $\lambda \in \Bbbk^*$, and q is a polynomial in λ with coefficients $b_i = f_i(a_1, \ldots, a_{n+1})$. Since $\Bbbk$ is infinite and $q(\lambda) = 0$, we have that $b_i = 0$ for each i. Therefore, $f_i(\lambda a_1, \ldots, \lambda a_{n+1}) = 0$, and $f_i(P) = 0$ for each i. The converse is clear. □

Thus, if

$$f = f_m + \cdots + f_d \in \Bbbk[x_1, \ldots, x_{n+1}],$$

where each f_i is an i-form, then $\mathrm{V_p}(f) = \mathrm{V_p}(f_m, \ldots, f_d)$ and if $f \in \mathrm{I_p}(Y)$ for some $Y \subseteq \mathbb{P}^n$, then $f_i \in \mathrm{I_p}(Y)$ for each i.

1.1.6 Proposition.

(i) *Every algebraic set in $\mathbb{P}^n$ is the zero set of a finite number of forms.*
(ii) *If $Y \subseteq \mathbb{P}^n$, then $\mathrm{I_p}(Y)$ is generated by homogeneous polynomials.*

Proof: The result is clear from the preceding discussion. □

This motivates the following definition.

1.1.7 Definition. An ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ is said to be *homogeneous* if whenever $f \in I$ then each homogeneous component of f is in I.

By the above, we see that $\mathrm{I_p}(Y)$ is homogeneous for any $Y \subseteq \mathbb{P}^n$. Moreover, one proves as in the affine case that $\mathrm{I_p}(Y)$ is radical. We then have a correspondence between projective algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals in $\Bbbk[x_1, \ldots, x_{n+1}]$. We will see that this is almost a one-to-one correspondence, but in order to make it one-to-one we have to exclude $\varnothing$ and the ideal of $\{(0, \ldots, 0)\}$. Let us begin by stating some properties of homogeneous ideals.

1.1.8 Proposition. *Let I and J be ideals in $\Bbbk[x_1, \ldots, x_{n+1}]$. Then:*

(i) *I is homogeneous if and only if I can be generated by homogeneous polynomials;*
(ii) *if I and J are homogeneous, then $I + J$, IJ, $I \cap J$, and $\sqrt{I}$ are homogeneous;*
(iii) *I is a homogeneous prime ideal if and only if whenever $f, g \in \Bbbk[x_1, \ldots, x_n]$ are forms such that $fg \in I$, then $f \in I$ or $g \in I$.*

PROOF:
(i) Exercise
(ii) Exercise.
(iii) The forward direction is clear. Let $f, g \in \Bbbk[x_1, \ldots, x_{n+1}]$ be such that $fg \in I$. Let $f = \sum_{i=m}^{d} f_i$ and $g = \sum_{i=m'}^{d'} g_j$, where each f_i is an i-form and each g_j is a j-form. Then

$$fg = f_m g_{m'} + \sum_{k>m+m'}^{d+d'} \left(\sum_{i+j=k} f_i g_i \right).$$

Since I is homogeneous, $f_m g_{m'} \in I$. Suppose, for now, that $f_m \notin I$, so that $g_{m'} \in I$. Then $f(g - g_{m'}) = (fg - fg_{m'}) \in I$. Since the homogeneous component of $f(g - g_{m'})$ of degree $m + m' + 1$ is $f_m g_{m'+1}$ and I is homogeneous, we have $f_m g_{m'+1} \in I$. And since $f_m \notin I$, this means that $g_{m'+1} \in I$ so that

$$f(g - g_{m'} - g_{m'+1}) = fg - fg_{m'} - fg_{m'+1} \in I.$$

Continuing this way, we see that if $f_m \notin I$, then $g_i \in I$ for all $i = m', \ldots, d'$, implying that $g \in I$. If both f_m and $g_{m'}$ are in I, proceed as above with $(f - f_m)(g - g_{m'})$. $\square$

1.1.9 Examples.
(i) $I = \langle x^2 \rangle$ and $J = \langle x^2, y \rangle$ are homogeneous in $\Bbbk[x, y]$.

(ii) $I = \langle x^2 + x \rangle$ is not homogeneous since $x \notin I$.

1.1.10 Definition. Let $\theta : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the standard projection, so that

$$\theta(x_1, \dots, x_{n+1}) = [x_1 : \cdots : x_{n+1}].$$

If $Y \subseteq \mathbb{P}^n$, the *affine cone* over Y is

$$\mathrm{C}(Y) = \theta^{-1}(Y) \cup \{(0, \dots, 0)\} \subseteq \mathbb{A}^{n+1}.$$

We note the following properties of the affine cone.

1.1.11 Proposition.
(i) *If $P \in \mathbb{P}^n$, then $\mathrm{C}(\{P\})$ is the line in $\mathbb{A}^{n+1}$ through the origin determined by P.*
(ii) $\mathrm{C}(\varnothing) = \{(0, \dots, 0)\}$.
(iii) $\mathrm{C}(Y_1 \cup Y_2) = \mathrm{C}(Y_1) \cup \mathrm{C}(y_2)$.
(iv) $\mathrm{C}(Y_1) = \mathrm{C}(Y_2)$ *if and only if* $Y_1 = Y_2$.
(v) *If $Y \subseteq \mathbb{P}^n$ is non-empty, then $\mathrm{I_p}(Y) = \mathrm{I_a}(\mathrm{C}(Y))$.*
(vi) *If $I \subseteq \Bbbk[x_1, \dots, x_{n+1}]$ is a homogeneous ideal such that $\mathrm{V_p}(I) \neq \varnothing$, then $\mathrm{C}(\mathrm{V_p}(I)) = \mathrm{V_a}(I)$. In particular, $C(Y) = \mathrm{V_a}(I)$ for some non-empty $Y \subseteq \mathbb{P}^n$ if and only if $Y = \mathrm{V_p}(I)$.*

One can use affine cones to compute $\mathrm{I_p}$ and $\mathrm{V_p}$ or to determine properties of $\mathrm{I_p}$ and $\mathrm{V_p}$.

1.1.12 Examples (of projective algebraic sets).

(i) $\mathrm{V_p}(I) = \mathrm{V_p}(1) = \varnothing$ for any ideal I such that $\mathrm{V_a}(I) = \{(0, \dots, 0)\}$.
(ii) $\mathbb{P}^n = \mathrm{V_p}(0)$.
(iii) If $P = [a : b] \in \mathbb{P}^1$, then $\{P\} = \mathrm{V_p}(bx - ay)$, since $\mathrm{C}(\{P\})$ is the line through $(0, 0)$ and (a, b) in $\mathbb{A}^2$, which is $\mathrm{V_a}(bx - ay)$. In general, if $P = [a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$ and a_i is a non-zero coordinate of P, then

$$\{P\} = \mathrm{V_p}(a_i x_1 - a_1 x_i, \dots, a_i x_{n+1} - a_{n+1} x_i).$$

(iv) If f is a homogeneous polynomial, then $Y = \mathrm{V_p}(f)$ is called a *hypersurface*.
(v) Let $Y = \mathrm{V_p}(x - y, x^2 - yz) \subset \mathbb{P}^2$. Then

$$\begin{aligned} C(Y) = \mathrm{V_a}(x - y, x^2 - yz) &= \mathrm{V_a}(x, y) \cup \mathrm{V_a}(x - y, x - z) \\ &= \{(0, 0, s) \mid s \in \Bbbk\} \cup \{(t, t, t) \mid t \in \Bbbk\}, \end{aligned}$$

so

$$Y = \mathrm{V_p}(x, y) \cup \mathrm{V_p}(x - y, x - z) = \{[0 : 0 : 1]\} \cup \{[1 : 1 : 1]\}.$$

1.1.13 Examples (of ideals).

(i) $\mathrm{I_p}(\mathbb{P}^n) = \langle 0 \rangle$.
(ii) $\mathrm{I_p}(\varnothing) = \langle 1 \rangle$.
(iii) If $P = [a_1 : \cdots : a_{n+1}] \in \mathbb{P}^n$ with $a_i \neq 0$, then

$$\begin{aligned}\mathrm{I_p}(\{P\}) &= \mathrm{I_a}(\mathrm{C}([a_1 : \cdots : a_{n+1}])) \\ &= \langle a_i x_1 - a_1 x_i, \ldots, a_i x_{n+1} - a_{n+1} x_i \rangle,\end{aligned}$$

since $\mathrm{C}([a_1 : \cdots : a_{n+1}]) = \mathrm{V_a}(a_i x_1 - a_1 x_i, \ldots, a_i x_{n+1} - a_{n+1} x_i)$.

Remark. In the projective case, $\langle x_1, \ldots, x_{n+1} \rangle$ is a homogeneous radical ideal other than $\langle 1 \rangle$ whose zero set in $\mathbb{P}^n$ is empty. We must therefore remove $\varnothing$ and $\langle x_1, \ldots, x_{n+1} \rangle$ from the one-to-one correspondence between projective algebraic sets and radical homogeneous ideals.

1.1.14 Proposition. *The union of two projective algebraic sets is a projective algebraic set. The intersection of any family of projective algebraic sets is a projective algebraic set. Moreover, $\varnothing$ and $\mathbb{P}^n$ are projective algebraic sets.*

Therefore, the projective algebraic subsets of $\mathbb{P}^n$ are the closed sets of a topology on $\mathbb{P}^n$.

1.1.15 Definition. The *Zariski topology* on $\mathbb{P}^n$ is the topology whose open sets are the complements of projective algebraic sets.

1.1.16 Examples.

(i) For each i, the hyperplane $H_i = \mathrm{V_p}(x_i)$ is a closed set and its complement $U_i = \mathbb{P}^n \setminus H_i$ is an open set in the Zariski topology. Therefore, $\{U_i\}_{i=1}^{n+1}$ is an open cover of $\mathbb{P}^n$.
(ii) We have seen that $\mathbb{A}^n$ can be identified with the open set U_{n+1} in $\mathbb{P}^n$. For any affine variety $X \subset \mathbb{A}^n$, we define the *projective closure* of X in $\mathbb{P}^n$ to be the smallest projective algebraic set containing X. For example, $Y = \mathrm{V_p}(y^2 z - x^3) \subset \mathbb{P}^2$ is the projective closure of $X = \mathrm{V_a}(y^2 - x^3) \subset \mathbb{A}^2$ since
$$\mathrm{V_p}(y^2 z - x^3) = \mathrm{V_a}(y^2 - x^3) \cup \{[0 : 1 : 0]\},$$
i.e., Y is the one-point compactification of X in $\mathbb{P}^2$.

1.1.17 Definition. A non-empty closed subset of $\mathbb{P}^n$ is *irreducible* if it cannot be expressed as the union of two proper closed subsets. A *projective (algebraic) variety* is an irreducible algebraic set in $\mathbb{P}^n$ equipped with the induced Zariski topology.

As in the affine case, we have the following result.

1.1.18 Proposition. *Let $Y \subseteq \mathbb{P}^n$ be a projective algebraic set. Then Y is irreducible if and only if $\mathrm{I_p}(Y)$ is prime.*

Proof: Let $f, g \in \Bbbk[x_1, \dots, x_{n+1}]$ be forms such that $fg \in \mathrm{I_p}(Y)$. Then $\mathrm{V_p}(f)$ and $\mathrm{V_p}(g)$ are projective algebraic sets and

$$Y = (Y \cap \mathrm{V_p}(f)) \cup (Y \cap \mathrm{V_p}(g)),$$

so by the irreducibility of Y, $Y = Y \cap \mathrm{V_p}(f)$ or $Y = Y \cap \mathrm{V_p}(g)$, implying that $f \in \mathrm{I_p}(Y)$ or $g \in \mathrm{I_p}(Y)$.

The reverse direction is as in the affine case. □

1.1.19 Proposition. *Let Y be a subset of $\mathbb{P}^n$. Then:*

(i) *Y is a projective algebraic set if and only if $\mathrm{C}(Y)$ is an affine algebraic set.*
(ii) *Y is an irreducible projective algebraic set if and only if $\mathrm{C}(Y)$ is an irreducible affine algebraic set.*
(iii) *If Y is algebraic, then it is the union of a finite number of irreducible projective algebraic sets.*

Proof:
(i) Y is algebraic if and only if $\mathrm{I_p}(Y) = \mathrm{I_a}(\mathrm{C}(Y))$ is radical, which happens if and only if $\mathrm{C}(Y)$ is algebraic.
(ii) Y is an irreducible algebraic set if and only if $\mathrm{I_p}(Y) = \mathrm{I_a}(\mathrm{C}(Y))$ is prime, which happens if and only if $\mathrm{C}(Y)$ an irreducible algebraic set.
(iii) If Y is algebraic, then so is $C(Y)$, which is the union of a finite number of irreducible affine algebraic sets. If $C(Y) = \widetilde{W_1} \cup \dots \cup \widetilde{W_n}$ with each $\widetilde{W_i}$ irreducible, then $Y = W_1 \cup \dots \cup W_n$ with $W_i = \mathrm{V_p}(\mathrm{I_a}(\widetilde{W_i}))$ irreducible for all i. □

Remark. One defines the (irredundant) decomposition of a projective algebraic set Y as in the affine case. This decomposition is unique up to a permutation of its irreducible components since $W_1 \cup \dots \cup W_m$ is the decomposition of Y if and only if $C(W_1) \cup \dots \cup C(W_n)$ is the decomposition of $C(Y)$ (exercise).

1.1.20 Examples.
(i) If f is a form in $\Bbbk[x_1, \dots, x_{n+1}]$, then

$$\begin{aligned} \mathrm{V_p}(f) \text{ is irreducible } &\iff \mathrm{C}(\mathrm{V_p}(f)) = \mathrm{V_a}(f) \text{ is irreducible} \\ &\iff f \text{ is irreducible.} \end{aligned}$$

(ii) Let $Y = \mathrm{V_p}(x^2 + y^2 + 2yz) \subset \mathbb{P}^2$, and let

$$\begin{aligned} f &= x^2 + y^2 + 2yz \\ &= z^2 \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 + 2\left(\frac{y}{z}\right) \right) \\ &= z^2 g\left(\frac{x}{z}, \frac{y}{z}\right), \end{aligned}$$

where $g(u,v) = u^2 + v^2 + 2v$ is irreducible. Thus f is irreducible and Y is irreducible.

(iii) Is $Y = \mathrm{V_p}(xz^3 + y^2z^2 - x^3z - x^2y^2)$ irreducible? This time

$$\begin{aligned} g(u,v) &= u + v^2 - u^3 - u^2v^2 \\ &= (u + v^2)(1 - u^2) \\ &= (u + v^2)(1 - u)(1 + u), \end{aligned}$$

so $f = xz^3 + y^2x^2 - x^3z - x^2y^2 = z^4 g(x/z, y/z) = (xz + y^2)(z - x)(z + x)$, and

$$Y = \mathrm{V_p}(f) = \mathrm{V_p}(xz + y^2) \cup \mathrm{V_p}(z - x) \cup \mathrm{V_p}(z + x),$$

which is the irreducible decomposition of Y since $xz + y^2$, $z - x$, and $z + x$ are irreducible.

1.1.21 Theorem (Projective Nullstellensatz). *Let $I \subseteq \Bbbk[t_1, \ldots, t_{n+1}]$ be a homogeneous ideal. Then:*

(i) $\mathrm{V_p}(I) = \varnothing$ *if and only if there exists $N \in \mathbb{N}$ such that I contains every form of degree at least N;*

(ii) *if* $\mathrm{V_p}(I) \neq \varnothing$ *then* $\mathrm{I_p}(\mathrm{V_p}(I)) = \sqrt{I}$.

PROOF:

(i) The following statements are equivalent:

$$\begin{aligned} \mathrm{V_p}(I) = \varnothing &\iff \mathrm{V_a}(I) = \varnothing \text{ or } \{(0, \ldots, 0)\} \\ &\iff \mathrm{V_a}(I) \subseteq \{(0, \ldots, 0)\} \\ &\iff \langle x_1, \ldots, x_{n+1} \rangle = \mathrm{I_a}(\{(0, \ldots, 0)\}) \subseteq \mathrm{I_a}(\mathrm{V_a}(I)) = \sqrt{I} \\ &\iff x_i^{m_i} \in I \text{ for some } m_i \in \mathbb{N}, \text{ for all } i \\ &\iff \text{ any form of degree at least } N \text{ is contained in } I \\ &\qquad \text{ for some } N \geq \max\{m_1, \ldots, m_{n+1}\}. \end{aligned}$$

(ii) $\mathrm{I_p}(\mathrm{V_p}(I)) = \mathrm{I_a}(C(\mathrm{V_p}(I)) = \mathrm{I_a}(\mathrm{V_a}(I)) = \sqrt{I}$, by the affine Nullstellensatz.
□

As a consequence of the projective Nullstellensatz, we have the following one-to-one correspondences:

$$(\text{non-empty algebraic sets in } \mathbb{P}^n) \quad \longleftrightarrow \quad \begin{pmatrix} \text{proper homogeneous} \\ \text{radical ideals} \\ I \neq \langle x_1, \ldots, x_{n+1} \rangle \end{pmatrix}$$

$$(\text{varieties in } \mathbb{P}^n) \quad \longleftrightarrow \quad \begin{pmatrix} \text{homogeneous prime} \\ \text{ideals } I \neq \langle x_1, \ldots, x_{n+1} \rangle \end{pmatrix}.$$

The empty set is usually thought of as corresponding to $\langle x_1, \ldots, x_{n+1} \rangle$.

1.2 Regular and Rational Functions

1.2.1 Definition. Let $I \subseteq \Bbbk[x_1, \ldots, x_{n+1}]$ be a homogeneous ideal. A residue class in $\Bbbk[x_1, \ldots, x_{n+1}]/I$ is said to be an *m-form* if it contains an m-form. In particular, 0-forms are *constants*.

1.2.2 Proposition. *Let $I \subseteq \Bbbk[x_1, \ldots, x_{n+1}]$ be a homogeneous ideal. Every $\overline{f} \in \Bbbk[x_1, \ldots, x_{n+1}]/I$ may be expressed uniquely as $\overline{f} = \overline{f}_0 + \cdots + \overline{f}_d$, where $d = \deg f$ and each $\overline{f}_i$ is an i-form.*

PROOF: Existence is clear, so we need only show uniqueness. Suppose that

$$\overline{f} = \overline{f_m} + \cdots + \overline{f_d} = \overline{g_{m'}} + \cdots + \overline{g_{d'}},$$

where each f_i is an i-form and each g_j is a j-form. Then

$$\sum_i (\overline{f_i} - \overline{g_i}) = 0,$$

where we set $f_i = 0$ for $i < m$ and $i > d$ and $g_i = 0$ for $i < m'$ and $i > d'$. Thus

$$\sum_i (f_i - g_i) \in I,$$

so by the homogeneity of I, $f_i - g_i \in I$ for all i, so

$$\overline{f_m} + \cdots + \overline{f_d} = \sum_i \overline{f_i} = \sum_i \overline{g_i} = \overline{g_{m'}} + \cdots + \overline{g_{d'}}. \qquad \square$$

1.2.3 Definition. Let $Y \subseteq \mathbb{P}^n$ be a projective variety, so that $\mathrm{I_p}(Y) = \mathrm{I_a}(C(Y))$ is prime and homogeneous. Then

$$\Gamma_H(Y) = \Bbbk[x_1, \ldots, x_{n+1}]/\mathrm{I_p}(Y) = \Gamma(C(Y))$$

is an integral domain, called the *homogeneous coordinate ring*.

Note that, unlike the case of affine coordinate rings, elements of $\Gamma_H(Y)$ cannot be considered functions unless they are constant. Indeed, $\overline{f} \in \Gamma_H(Y)$ defines a function on Y if and only if $f(\lambda x_1, \ldots, \lambda x_{n+1}) = f(x_1, \ldots, x_{n+1})$ for all $\lambda \in \Bbbk^*$. But if $f = f_0 + \cdots + f_d$ is the decomposition of f into forms, then this happens if and only if

$$\begin{aligned} f(x_1, \ldots, x_{n+1}) &= f(\lambda x_1, \ldots, \lambda x_{n+1}) \\ &= f_0(x_1, \ldots, x_{n+1}) + \lambda f_1(x_1, \ldots, x_{n+1}) + \cdots + \lambda^d f_d(x_1, \ldots, x_{n+1}), \end{aligned}$$

for all $\lambda \in \Bbbk^*$, which can only happen if f is constant on Y, so that $\overline{f}$ is constant in $\Gamma_H(Y)$.

1.2.4 Definition. Let $Y \subseteq \mathbb{P}^n$ be a projective variety. The field of fractions of $\Gamma_H(Y)$ is denoted by $\Bbbk_H(Y)$, and is called the *homogeneous function field*.

Note that $\Bbbk_H(Y) = \Bbbk(C(Y))$. But the only elements of $\Bbbk_H(Y)$ that define functions on Y are of the form $\overline{f}/\overline{g}$ with $\overline{f}, \overline{g} \in \Gamma_H(Y)$ forms of the same degree and $\overline{g} \neq 0$. This is because if $\overline{f} = \overline{f}_m + \cdots + \overline{f}_d$ and $\overline{g} = \overline{g}_{m'} + \cdots + \overline{g}_{d'}$, then

$$\begin{aligned} \frac{f(\lambda x_1, \ldots, \lambda x_{n+1})}{g(\lambda x_1, \ldots, \lambda x_{n+1})} &= \frac{\lambda^m f_m(x_1, \ldots, x_{n+1}) + \cdots + \lambda^d f_d(x_1, \ldots, x_{n+1})}{\lambda^{m'} g_{m'}(x_1, \ldots, x_{n+1}) + \cdots + \lambda^{d'} g_{d'}(x_1, \ldots, x_{n+1})} \\ &= \frac{f(x_1, \ldots, x_{n+1})}{g(x_1, \ldots, x_{n+1})} \end{aligned}$$

for all $\lambda \in \Bbbk^*$ if and only if $m = m'$ and $f = f_m, g = g_m$ on Y, so that $\overline{f} = \overline{f}_m$ and $\overline{g} = \overline{g}_m$ are forms of the same degree. We then define

$$\Bbbk(Y) = \left\{ \frac{\overline{f}}{\overline{g}} \mid \overline{f}, \overline{g} \in \Gamma_H(Y) \text{ are forms of the same degree} \right\},$$

which is the *function field* of Y, whose elements are called *rational functions* on Y. We have

$$\Bbbk \subseteq \Bbbk(Y) \subseteq \Bbbk_H(Y) = \Bbbk(C(Y)),$$

but $\Gamma_H(Y) \not\subseteq \Bbbk(Y)$ in general.

1.2.5 Definition. If $p \in Y$ and $z \in \Bbbk(Y)$, we say that z is *regular at p* (or *defined at p*) if there exist forms $\overline{f}, \overline{g} \in \Gamma_H(Y)$ of the same degree such that $g(p) \neq 0$ and $z = \overline{f}/\overline{g}$, in which case $z(p) = f(p)/g(p)$ is the *value of f at p*. The set of points where z is not defined is called its *pole set*.

1.2.6 Proposition. *Let $Y \subseteq \mathbb{P}^n$ be a projective variety. Then the pole set of any rational function on Y is an algebraic subset of Y.*

Proof: The pole set of $z \in \Bbbk(Y)$ is the intersection of the algebraic sets $\mathrm{V_p}(g) \cap Y$, taken over all forms g for which there is a form f such that $z = \overline{f}/\overline{g}$. Hence it is algebraic. $\square$

1.2.7 Definition. Let Y be a projective variety, and let p be a point in Y. Then

$$\mathcal{O}_p(Y) = \{z \in \Bbbk(Y) \mid z \text{ is regular at } p\} \subseteq \Bbbk(Y)$$

is the *local ring of Y at p*,

$$\mathrm{M}_p(Y) = \{z \in \mathcal{O}_p(Y) \mid z(p) = 0\}$$

is the *maximal ideal of Y at p*, and

$$\mathcal{O}(Y) = \bigcap_{p \in Y} \mathcal{O}_p(Y)$$

is the *ring of regular functions on Y*.

As in the affine case, $\mathcal{O}_p(Y)$ is a local ring and $\mathrm{M}_p(Y)$ is its unique maximal ideal. However, in contrast to the affine case, $\mathcal{O}(Y)$ is not isomorphic to $\Gamma_H(Y)$.

1.2.8 Proposition. *Let Y be a projective variety. Then $\mathcal{O}(Y) = \Bbbk$.*

PROOF: $\mathcal{O}(Y) \subseteq \mathcal{O}(C(Y)) = \Gamma(C(Y)) = \Gamma_H(Y)$ and the only functions in $\Gamma_H(Y)$ are the constants. □

Nonetheless, one proves as in the affine case that if two rational functions on Y are equal on an open set $U \subseteq Y$, then they are equal on Y. In particular, one has the following.

1.2.9 Proposition. *Let $Y \subseteq \mathbb{P}^n$ be a projective variety. Then*

$$\Bbbk(Y) \simeq \Bbbk(Y \cap U_i)$$

are isomorphic as $\Bbbk$-algebras for all i, where U_i is the affine open subset of $\mathbb{P}^n$ given by $x_i \neq 0$. Moreover, if $p \in Y \cap U_i$, then

$$\mathcal{O}_p(Y) \simeq \mathcal{O}_p(Y \cap U_i)$$

as $\Bbbk$-algebras.

PROOF: Define $\Phi : \Bbbk(Y) \to \Bbbk(Y \cap U_i)$ by

$$\Phi\left(\frac{\overline{f}}{\overline{g}}\right) = \overline{\frac{f(x_1, \ldots, 1, \ldots, x_{n+1})}{g(x_1, \ldots, 1, \ldots, x_{n+1})}},$$

and $\Psi : \Bbbk(Y \cap U_i) \to \Bbbk(Y)$ by

$$\Psi\left(\frac{\overline{a}}{\overline{b}}\right) = \frac{\overline{x_i}^d \overline{a}(x_1/x_i, \ldots, x_{n+1}/x_i)}{\overline{x_i}^d \overline{b}(x_1/x_i, \ldots, x_{n+1}/x_i)},$$

where $d = \max\{\deg a, \deg b\}$. It is then easy to check that these are $\Bbbk$-algebra homomorphisms that are mutual inverses. The second statement is a direct consequence of the first. □

From the above proposition, we see that the local properties of a projective variety Y can be completely described in terms of its affine pieces $Y \cap U_i$. This is for instance the case with dimension and smoothness.

1.2.10 Definition. Let $Y \subseteq \mathbb{P}^n$ be a variety. The *dimension of* Y is defined as $\dim Y := tr.\deg_{\Bbbk}(\Bbbk(Y))$.

By the above, dimension is a local property as $\dim Y = \dim(Y \cap U_i)$ for all i. In particular,

$$\dim \mathbb{P}^n = \dim \mathbb{A}^n = n.$$

Moreover, projective varieties of dimensions 1, 2, 3, are called *curves, surfaces, 3-folds* etc... In particular, if Y is the zero set in $\mathbb{P}^n$ set of a single irreducible form, then Y has dimension $n-1$ and is called a *hypersurface*.

Remark. One can also show that $\dim Y = \dim C(Y) - 1$ (exercise).

Furthermore, one defines smoothness as follows.

1.2.11 Definition. Let $Y \subseteq \mathbb{P}^n$ be an r-dimensional projective variety and $p \in Y$. The $\Bbbk$-vector space

$$T_p(Y) := \left(M_p(Y)/(M_p(Y))^2\right)^*$$

is called the *Zariski tangent space of* Y *at* p, and Y is said to be *smooth at* p if and only if $\dim_{\Bbbk} T_p(Y) = r$. Otherwise, p is called *singular*. Moreover, Y is called *smooth* if it is smooth at every point.

Remark. Clearly, Y is smooth at p if and only if $Y \cap U_i$ is smooth at p since $\mathcal{O}_p(Y) \simeq \mathcal{O}_p(Y \cap U_i)$ for any affine open subset U_i containing p. To check the smoothness of Y at p one then just has to compute the rank of the Jacobian of the polynomials giving $Y \cap U_i \subseteq \mathbb{A}^n$ at p. However, it is in practice not necessary to consider the restriction of Y to $Y \cap U_i$ since one can show, as in the affine case, the following.

If $Y = \mathrm{V_p}(f_1, \dots, f_s) \subseteq \mathbb{P}^n$ for some forms $f_1, \dots, f_s \in \Bbbk[x_1, \dots, x_n]$, then

$$T_p(Y) = \ker(\mathrm{Jac}(f_1, \dots, f_s)(p)),$$

so that Y is smooth at p if and only if $\mathrm{Jac}(f_1, \dots, f_s)(p)$ has rank $n - \dim Y$.

The proof is left to the reader as an exercise.

1.2.12 Example. Consider the projective plane curve $Y = \mathrm{V_p}(f) \subset \mathbb{P}^2$ where $f = axy + bxz + cyz \in \Bbbk[x, y, z]$. Then Y is smooth if and only if $a, b, c \neq 0$. Indeed, $\mathrm{Jac}(f) = (ay + bz, ax + cz, bx + cy)$. If one of the a, b, c is zero, say a, then $\mathrm{Jac}(f)$ has rank 0 at $[x : y : z] = [c, -b, 0] \in Y$, and Y has singular points. But if $a, b, c \neq 0$, then $\mathrm{Jac}(f)$ has rank 0 if and only if $(x, y, z) = (0, 0, 0)$, which does not correspond to a point on Y, implying that Y is smooth.

Finally, as in the affine case, we have:

1.2.13 Proposition. *A projective curve Y is smooth at p if and only if $\mathcal{O}_p(Y)$ is a DVR.*

Proof: Y is smooth at p if and only if $M_p(Y)/(M_p(Y))^2$ is a 1-dimensional $\Bbbk$-vector space, which happens if and only if $M_p(Y)$ is principal. □

1.3 Regular and Rational Maps

1.3.1 Definition. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties. A map $\varphi : X \to Y$ is called *rational* if it can be written as

$$\varphi(x_1 : \cdots : x_{n+1}) = [F_1(x_1, \ldots, x_{n+1}) : \cdots : F_{m+1}(x_1, \ldots, x_{n+1})]$$

for some forms $F_1, \ldots, F_{m+1} \in \Bbbk[x_1, \ldots, x_{n+1}]$ of the same degree. Moreover, φ is said to be *regular*, or defined, at $p \in X$ if it can be represented by forms $F_1, \ldots, F_{m+1}$ that do not vanish simultaneously at p. If φ is not defined at p, then p is said to be a *pole* of φ. If φ is regular at every point in X, then it is called a *regular map*.

A rational map $\varphi : X \to Y$ that has a rational inverse $\varphi^{-1} : Y \to X$ is called a *birational equivalence*, in which case we say that X and Y are *birational* and write $X \sim Y$. If φ and φ^{-1} are both regular, then φ is called a *isomorphism*, in which case we say that X and Y are *isomorphic* and write $X \cong Y$.

Remarks. (i) One can also define rational maps $\varphi : X \to Y$ as maps that can be written as

$$\varphi(x_1 : \cdots : x_{n+1}) = [h_1(x_1, \ldots, x_{n+1}) : \cdots : h_{m+1}(x_1, \ldots, x_{n+1})]$$

for some rational functions $h_1, \ldots, h_{m+1} \in \Bbbk(Y)$. But each $h_i = \overline{f}_i/\overline{g}_i$ with f_i and g_i forms of the same degree in $\Bbbk[x_1, \ldots, x_{n+1}]$. By clearing denominators, we can write

$$\varphi(x_1 : \cdots : x_{n+1}) = [F_1(x_1, \ldots, x_{n+1}) : \cdots : F_{m+1}(x_1, \ldots, x_{n+1})],$$

where $F_i := g_1 \cdots g_{i-1} f_i g_{i+1} \cdots g_{m+1}$ are forms of the same degree. The two definitions of rational map are therefore equivalent.

(ii) As in the affine case, we can define the pullback of a rational map, and we have that $X \sim Y$ if and only if $\Bbbk(X)$ and $\Bbbk(Y)$ are isomorphic as $\Bbbk$-algebras. Consequently, dimension is preserved under birational equivalences.

(iii) As in the affine case, smoothness is preserved under isomorphisms.

1.3.2 Examples.

(i) Any rational function $h : X \to \Bbbk$ can be considered as a rational map φ from X to $\mathbb{P}^1$. If $h = \overline{f}/\overline{g}$, set

$$\varphi(x_1 : \cdots : x_{n+1}) = [f(x_1, \ldots, x_{n+1}) : g(x_1, \ldots, x_{n+1})].$$

(ii) Any invertible matrix $A \in GL(n+1, \mathbb{k})$ defines an isomorphism $T : \mathbb{P}^n \to \mathbb{P}^n, x \mapsto Ax$, called a *projective coordinate change*, since matrix multiplication commutes with scalar multiplication in $\mathbb{A}^{n+1}$ and therefore descends to the quotient $\mathbb{P}^n = (\mathbb{A}^{n+1} \backslash \{0\})/\mathbb{k}^*$.

(iii) Any hyperplane $H = \mathrm{V_p}(a_1 x_1 + \cdots + a_{n+1} x_{n+1})$ in $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^{n-1}$ since it can be mapped isomorphically onto the hyperplane at infinity $H_\infty = \mathbb{P}^{n-1}$ under an appropriate projective coordinate change.

(iv) Let $Y = \mathrm{V_p}(xz - y^2) \subset \mathbb{P}^2$, and define $\varphi : \mathbb{P}^1 \to Y$ by

$$\varphi(u : v) = [u^2 : uv : v^2].$$

Since u and v can not be simultaneously zero, u^2, uv, and v^2 can not be simultaneously zero, so we see that φ is regular at every point. Also, φ has a regular inverse defined by

$$\varphi^{-1}(x : y : z) = \begin{cases} [x : y] & \text{if } x \neq 0, \\ [y : z] & \text{if } z \neq 0. \end{cases}$$

We only need to verify that φ^{-1} is well-defined. Note that if $x, z \neq 0$ on Y, then $y \neq 0$, in which case

$$[x : y] = [xz : yz] = [y^2 : yz] = [y : z].$$

So φ^{-1} is well-defined on Y, showing that Y is isomorphic to $\mathbb{P}^1$.

More generally, one can show that if $Y = \mathrm{V_p}(f) \subset \mathbb{P}^2$ is the zero set of an irreducible 2-form $f \in \mathbb{k}[x, y, z]$, then Y is isomorphic to $\mathbb{P}^1$ (exercise), implying it is a smooth curve.

(v) Let $Y = \mathrm{V_p}(y^2 z - x^3) \subset \mathbb{P}^2$, and define $\varphi : \mathbb{P}^1 \to Y$ by

$$\varphi(u : v) = [u^2 v : u^3 : v^3].$$

As in the previous example, it is easy to see that φ is regular at every point. However, it can not have a regular inverse, as then restricting to U_z would imply that $\mathrm{V_a}(y^2 - x^3)$ is isomorphic $\mathbb{A}^1$, which we know is not true. Nonetheless, φ does have a rational inverse $\varphi^{-1} : Y \to \mathbb{P}^1$ given by

$$\varphi^{-1}(x : y : z) = [y : x],$$

so φ is a birational equivalence, showing that Y is birational to $\mathbb{P}^1$. This is to be expected as $Y \cap U_z = \mathrm{V_a}(y^2 - x^3)$ and

$$Y = \mathrm{V_a}(y^2 - x^3) \cup \{[0 : 1 : 0]\},$$

and we have seen that $\mathrm{V_a}(y^2 - x^3)$ is birational to $\mathbb{A}^1$.

The above examples motivate the following definition.

1.3.3 Definition. Let Y be a projective variety. We say that Y is *rational* if it is birational to $\mathbb{P}^n$ for some n.

We will show the existence of non-rational varieties in the next chapter. We end this chapter with a few facts about projective curves.

1.3.4 Proposition. *Let C be a projective curve in $\mathbb{P}^n$, and let $\varphi : C \to \mathbb{P}^m$ be a rational map. Then φ is regular at every smooth point of C.*

Proof: Let p be a smooth point of C. Let $F_1, \ldots, F_{m+1} \in \Bbbk[x_1, \ldots, x_{n+1}]$ be forms such that

$$\varphi(x_1 : \cdots : x_{n+1}) = [F_1(x_1, \ldots, x_{n+1}) : \cdots : F_{m+1}(x_1, \ldots, x_{n+1})].$$

Let $t \in \mathcal{O}_p(C)$ be a local parameter. Then each F_i can be written in the form

$$F_i = t^{k_i} u_i$$

for some $k_i \in \mathbb{Z}$ and unit $u_i \in \mathcal{O}_p(C)$. After a possible change of coordinates in $\mathbb{P}^m$, we may assume that $k_1 \leq k_2 \leq \cdots \leq k_{m+1}$. Then

$$\begin{aligned}\varphi(x_1 : \cdots : x_{n+1}) &= [F_1(x_1, \ldots, x_{n+1}) : \cdots : F_{m+1}(x_1, \ldots, x_{n+1})] \\ &= [t^{k_1} u_1(x_1, \ldots, x_{n+1}) : \cdots : t^{k_{m+1}} u_{m+1}(x_1, \ldots, x_{n+1})] \\ &= [u_1 : t^{k_2 - k_1} u_2 : \cdots : t^{k_{m+1} - k_1} u_{m+1}].\end{aligned}$$

The first component is non-zero at p since u_1 is a unit in $\mathcal{O}_p(C)$, and each of the components is regular since $k_1 \leq k_2 \leq \cdots \leq k_{m+1}$ and each u_i is a unit in $\mathcal{O}_p(C)$. Therefore, φ is regular at p. □

1.3.5 Corollary. *Let C and C' be smooth projective curves and let $\varphi : C \to C'$ be a birational equivalence. Then φ is an isomorphism.*

Proof: Since C and C' are smooth, the preceding proposition implies that both φ and φ^{-1} are regular everywhere. Therefore, φ is an isomorphism. □

1.3.6 Corollary. *A smooth rational projective curve is isomorphic to $\mathbb{P}^1$.*

We have seen above that projective plane curves $Y = \mathrm{V_p}(f) \subset \mathbb{P}^2$ given by 1-forms or irreducible 2-forms $f \in \Bbbk[x, y, z]$, which are called *lines* or *irreducible conics*, respectively, are isomorphic to $\mathbb{P}^1$ and therefore rational. We will see in the next chapter that *smooth plane cubics*, which are projective plane curves given by 3-forms, cannot be rational because they cannot be isomorphic to $\mathbb{P}^1$; this will be done using divisors. Smooth plane cubics are the simplest examples of projective varieties that are not rational.

Chapter 2

Projective Plane Curves

2.1 Projective Plane Curves

2.1.1 Definition. A projective plane curve is an equivalence class of non-constant forms in $\Bbbk[x, y, z]$, where $f \sim g$ if and only if $f = \alpha g$ for some $\alpha \in \Bbbk^*$. The *degree* of a curve is defined to be the degree of the defining form. Curves of degrees 1, 2, 3, and 4 are called lines, conics, cubics, and quartics, respectively.

If f is irreducible, then the projective curve given by f is the projective variety $\mathrm{V_p}(f)$ in $\mathbb{P}^2$. Local properties of a curve $C = \mathrm{V_p}(f)$ are given by restricting C to the affine open sets U_x, U_y, and U_z:

$$\begin{aligned} C \cap U_x &= \mathrm{V_a}(f(1, y, z)) \\ C \cap U_y &= \mathrm{V_a}(f(x, 1, z)) \\ C \cap U_z &= \mathrm{V_a}(f(x, y, 1)). \end{aligned}$$

For example, if $p = [x_0 : y_0 : 1] \in C \cap U_z$, then C is smooth at p if and only if $C \cap U_z$ is smooth at (x_0, y_0) and the *multiplicity* of C at p is defined to be

$$\mathrm{m}_p(f) = \mathrm{m}_{(x_0,y_0)}(f(x, y, 1))$$

so that p is *singular* if and only if $\mathrm{m}_p(f) \geq 2$. Intersection multiplicity is similarly defined as the usual affine intersection multiplicity on any affine open set containing the points. Using Proposition 1.2.9, it is easy to check that each of these definitions is independent of the affine open set chosen.

Remark.

How does one find intersection points of projective plane curves? If $C = \mathrm{V_p}(f)$ and $D = \mathrm{V_p}(g)$, to find $C \cap D$ solve the two systems

$$\begin{aligned} f(x, y, z) &= 0 \\ g(x, y, z) &= 0 \\ z &= 0, \end{aligned}$$

and

$$\begin{aligned} f(x, y, z) &= 0 \\ g(x, y, z) &= 0 \\ z &= 1, \end{aligned}$$

to find the points on the line at infinity and the points on U_z respectively, and discard $(0, 0, 0)$. For example, if $C = \mathrm{V}_{\mathrm{p}}(f)$, where $f = x^2 - y^2 + xz$, and $D = \mathrm{V}_{\mathrm{p}}(g)$, where $g = x + y$, the points at infinity are given by solving the system

$$\begin{aligned} x^2 - y^2 + xz &= 0 \\ x + y &= 0 \\ z &= 0, \end{aligned}$$

which has the solutions $y = -x$ and $z = 0$, which represent the single point $[-1 : 1 : 0]$ in $\mathbb{P}^2$. The points on U_z are given by solving the system

$$\begin{aligned} x^2 - y^2 + xz &= 0 \\ x + y &= 0 \\ z &= 1, \end{aligned}$$

which has the solutions $y = -x = 0$ and $z = 1$, which corresponds to the point $[0 : 0 : 1] \in \mathbb{P}^2$. Therefore, there are only two points of intersection, $[1 : 1 : 0]$ and $[0 : 0 : 1]$.

2.2 Bézout's Theorem

This section will be devoted to the proof of the following theorem and some of its corollaries.

2.2.1 Theorem (Bézout). *Let $C = \mathrm{V}_{\mathrm{p}}(f)$ and $D = \mathrm{V}_{\mathrm{p}}(g)$ be projective plane curves that do not have a common component. Then, if C has degree m and D has degree n, C and D intersect in mn points counting multiplicity.*

Suppose that C and D are given by the forms f and g respectively, so that $\deg(f) = m$ and $\deg(g) = n$. Then, since C and D do not have a common component, f and g can not have a common factor; moreovoer, C and D intersect in a finite set of points. We may therefore assume, after an appropriate projective change of coordinates, that none of the intersection points lie on the line at infinity.

Indeed, since C and D intersect in a finite set of points, we can find a line in $\mathbb{P}^2$ that does not contain any of the intersection points. This line is then given by a 1-form $ax + by + cz \in \Bbbk[x, y, z]$. At least one of the constants a, b, c must be non-zero. After a simple projective coordinate change, we may assume that $a \neq 0$. The matrix

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$$

is then invertible and corresponds to the projective coordinate change

$$[x : y : z] \mapsto [z : y : ax + by + cz] = [u : v : w].$$

This transformation then takes the line $ax + by + cz = 0$ to the line at infinity, i.e. the line $w = 0$.

We therefore assume that C and D do not intersect on the line at infinity given by $z = 0$, so that f and g do not have any common zeros on $z = 0$. This implies, in particular, that z does not divide f or g. Indeed, suppose that z divides f. Then $f = zf'$ for some form $f' \in \mathbb{k}[x, y, z]$. Consider the restriction of g to oints of the form $[x : 1 : 0]$. Then $g(x, 1, 0)$ must have at least one zero since $\mathbb{k}$ is algebraically closed. If $[x_0 : 1 : 0]$ is such a zero, then $g(x_0, 1, 0) = 0$ and $f(x_0, 1, 0) = f(x_0, 1, 0)$, since $f = zf'$, contradicting the fact that f and g do not have any common zeros at infinity. Therefore, our assumption that z divides f is false. A symmetric argument establishes that z also does not divide g. These facts are crucial for the proof of Bézout's Theorem.

Now, since C and D do not intersect at infinity, we have that $C \cap D \subseteq U_Z$. We therefore only have to prove that C and D intersect in mn points in U_z, counting multiplicity. Also, recall that

$$\begin{aligned} \sum_{p \in C \cap D} \mathrm{I}(p, C \cap D) &= \sum_{p \in C \cap D} \mathrm{I}(p, \mathrm{V_a}(f(x, y, 1)) \cap \mathrm{V_a}(g(x, y, 1))) \\ &= \dim_{\mathbb{k}}(\mathbb{k}[x, y]/\langle f(x, y, 1), g(x, y, 1)\rangle) \\ &= \dim_{\mathbb{k}}(\Gamma_*) \end{aligned}$$

where

$$\Gamma_* = \mathbb{k}[x, y]/\langle f(x, y, 1), g(x, y, 1)\rangle.$$

We thus have to prove that $\dim_{\mathbb{k}}(\Gamma_*) = mn$. We will do this by showing that $\Gamma_* \cong \Gamma_d$, where Γ_d is the $\mathbb{k}$-vector space of d-forms in $\Gamma = \mathbb{k}[x, y, z]/\langle f, g\rangle$, whenever $d \geq m + n$, and then we will show that $\dim_{\mathbb{k}}(\Gamma_d) = mn$. Before we continue, we will fix some more notation. Let $R = \mathbb{k}[x, y, z]$, and let R_d be the $\mathbb{k}$-vector space of all d-forms in R.

Remarks.

(i) Let $\langle f, g\rangle_d$ be the set of all d-forms in $\langle f, g\rangle$. Then

$$\Gamma_d \cong R_d/\langle f, g\rangle_d.$$

Indeed, if $\overline{h_1}, \overline{h_2} \in \Gamma_d$, one can choose $h_1, h_2 \in R_d$. So, if $\overline{h_1} = \overline{h_2}$, then

$$(h_1 - h_2) \in R_d \cap \langle f, g\rangle = \langle f, g\rangle_d.$$

(ii) If F is a polynomial of degree d in $\Bbbk[x,y]$, then $z^d F(x/z, y/z)$ is a d-form in R.
(iii) If F is a d-form in R, then $F = z^d F(x/z, y/z, 1)$.

2.2.2 Proposition. *With the above notation,*

(i) $\dim_{\Bbbk}(R_d) = (d+1)(d+2)/2$,
(ii) $\dim_{\Bbbk}(\Gamma_d) = mn$ *whenever* $d \geq m+n$.

PROOF:
(i) There are $(d+1)(d+2)/2$ monomials $x^r y^s z^t$ of degree $d = r+s+t$ in R. Moreover, any d-form in R is a linear combination of monomials of degree d.
(ii) Since $\Gamma_d \cong R_d/\langle f, g\rangle_d$, it ise nough to show that

$$\dim_{\Bbbk}(\langle f, g\rangle_d) = \dim_{\Bbbk}(R_d) - mn.$$

Define $\varphi : R_{d-m} \times R_{d-n} \to \langle f, g\rangle_d$ by $\varphi(a,b) = af + bg$. Then

$$\ker(\varphi) = \{(a,b) \in R_{d-m} \times R_{d-n} \mid af + bg = 0\}.$$

If $af + bg = 0$, then $af = -bg$, so since f and g do not have common factors, this implies that

$$a = gc \qquad \text{and} \qquad b = -fc$$

for some $c \in R_{d-m-n}$. Define $\psi : R_{d-m-n} \to \ker(\varphi)$ by $\psi(c) = gcf + (-fc)g$. Clearly, ψ is $\Bbbk$-linear and injective, and the above argument establishes that it is surjective. Hence $\ker(\varphi) \cong R_{d-m-n}$, so that

$$\langle f, g\rangle_d \cong (R_{d-m} \times R_{d-n})/R_{d-m-n}.$$

Therefore,

$$\begin{aligned}
\dim_{\Bbbk}(\langle f, g\rangle_d) &= \dim_{\Bbbk}(R_{d-m}) + \dim_{\Bbbk}(R_{d-n}) - \dim_{\Bbbk}(R_{d-m-n}) \\
&= \frac{(d+1)(d+2)}{2} - mn \\
&= \dim_{\Bbbk}(R_d) - mn,
\end{aligned}$$

as desired. □

2.2.3 Proposition. *With the above notation,*

(i) *the map* $\varphi : \Gamma_d \to \Gamma_*$ *given by* $\varphi(\overline{h}) = \overline{h(x,y,1)}$ *is a well-defined injective* $\Bbbk$*-linear map that is surjective for all* $d \geq m+n$,
(ii) $\dim_{\Bbbk}(\Gamma_*) = mn$.

PROOF:

(i) For any $r \in \mathbb{N}$, define $\alpha : \Gamma \to \Gamma$ by $\alpha(\overline{h}) = \overline{z^r h}$. We claim that α is injective. It is enough to show the case when $r = 1$, i.e. that if $\overline{zh} = 0$, then $\overline{h} = 0$. Note that since z does not divide either f or g, $f(x, y, 0)$ and $g(x, y, 0)$ are both not identically zero. Moreover, since f and g do not have a common factor, $f(x, y, 0)$ and $g(x, y, 0)$ are relatively prime. We will use these facts in the proof. Suppose that $\overline{zh} = 0$. Then there exist $a, b \in R$ such that $zh = af + bg$. Substituting $z = 0$, we have that

$$0 = a(x, y, 0)f(x, y, 0) + b(x, y, 0)g(x, y, 0),$$

so

$$a(x, y, 0)f(x, y, 0) = -b(x, y, 0)g(x, y, 0),$$

and since $f(x, y, 0)$ and $g(x, y, 0)$ are relatively prime, there exists a $c \in \mathbb{k}[x, y]$ such that

$$a(x, y, 0) = -cf(x, y, 0) \qquad \text{and} \qquad b(x, y, 0) = cg(x, y, 0).$$

Let

$$a_1 = a + cg \qquad \text{and} \qquad b_1 = b - cf.$$

Then

$$a_1 f + b_1 g = af + bg = zh,$$

and

$$a_1(x, y, 0) = b_1(x, y, 0) = 0.$$

Hence there exist $a', b' \in R$ such that

$$a_1 = za' \qquad \text{and} \qquad b_1 = zb',$$

implying that

$$zh = a_1 f + b_1 g = z(a' f + b' g),$$

so $h = a' f + b' g$ and $\overline{h} = 0$, proving the injectivity of α. Consider the induced map $\alpha : \Gamma_d \to \Gamma_{d+r}$. Then α is injective, but it is also surjective if $d \geq m + n$, since by part (ii) Proposition 2.2.2 Γ_t is a $\mathbb{k}$-vector space of dimension mn for all $t \geq m + n$. Define a $\mathbb{k}$-linear map $\varphi : \Gamma_d \to \Gamma_*$ by $\varphi(\overline{h}) = \overline{h(x, y, 1)}$. We must first check that φ is well-defined. Indeed, if $\overline{h_1} = \overline{h_2}$ in Γ_d, then

$$h_1 = h_2 = af + bg,$$

so that

$$h(x, y, 1) = h_2(x, y, 1) + a(x, y, 1)f(x, y, 1) + b(x, y, 1)g(x, y, 1),$$

implying that $\overline{h_1(x, y, 1)} = \overline{h_2(x, y, 1)}$ in Γ_*. We will now show that φ is a bijection. Suppose $\overline{h} \in \Gamma_d$, where h is a d-form, and $\overline{h(x, y, 1)} = 0$. Then

$$h(x, y, 1) = a(x, y)f(x, y, 1) + b(x, y)g(x, y, 1),$$

so for sufficiently large $t \geq d$, $z^{t-m}a(x/z, y/z)$ and $z^{t-n}b(x/z, y/z)$ are both forms and

$$z^{t-d}h = z^t h(x/z, y/z, 1) = af + bg,$$

so $\overline{z^{t-d}h} = 0$ in Γ_t, and $\overline{h} = 0$, because the map α constructed early was injective. Fix $\overline{Q} \in \Gamma_*$. Let $s = \deg(Q)$ and let $t = \max s, d$. Then $q = z^t Q(x/z, y/z)$ is a t-form, so $q \in \Gamma_t \cong \Gamma_d$, since $t \geq d \geq m + n$. Hence $\overline{q} = \overline{z^{t-d}h}$ for some $h \in \Gamma_d$. Therefore,

$$\overline{Q(x, y)} = \overline{q(x, y, 1)} = \overline{h(x, y, 1)} = \varphi(\overline{h}),$$

showing that φ is surjective.

(ii) By (i), Γ_d and Γ_* are isomorphic as $\Bbbk$-vector spaces whenever $d \geq m+n$. By part (ii) of Proposition 2.2.2, $\dim_{\Bbbk}(\Gamma_d) = mn$, so $\dim_{\Bbbk}(\Gamma_*) = mn$. □

Therefore, by the discussion preceding these propositions, we have established Bézout's Theorem. There are a number of immediate corollaries.

2.2.4 Corollary. *Let $C = \mathrm{V_p}(f)$ and $D = \mathrm{V_p}(g)$ be projective plane curves, of degrees m and n respectively. If C and D have no common component, then*

$$\sum_{q \in C \cap D} \mathrm{m}_q(f)\,\mathrm{m}_q(g) \leq \sum_{q \in C \cap D} \mathrm{I}(q, C \cap D) = mn.$$

2.2.5 Corollary. *Let $C = \mathrm{V_p}(f)$ and $D = \mathrm{V_p}(g)$ be projective plane curves, of degrees m and n respectively. If C and D intersect in mn distinct points, then these points are smooth points on C and D.*

Proof: In this case, the previous corollary implies that $\mathrm{m}_q(f) = \mathrm{m}_q(g) = 1$, which is equivalent to q being a smooth point on both C and D. □

2.2.6 Corollary. *Let $C = \mathrm{V_p}(f)$ and $D = \mathrm{V_p}(g)$ be projective plane curves, of degrees m and n respectively. if C and D intersect in more than mn points, counting multiplicity, then they have a common component.*

We continue with some less obvious applications of Bézout's Theorem.

2.2.7 Proposition. *Any smooth projective plane curve is irreducible.*

Proof: Suppose instead that the smooth projective plane curve $C = \mathrm{V_p}(f)$ is reducible, i.e. that $f = ab$ for some forms $a, b \in \Bbbk[x, y, z]$. By Bézout's Theorem,

$$\mathrm{V_p}(a) \cap \mathrm{V_p}(b) \neq \varnothing.$$

Fix $q \in \mathrm{V_p}(a) \cap \mathrm{V_p}(b)$. Then, since $f = ab$,

$$\mathrm{m}_q(f) = \mathrm{m}_q(ab) = \mathrm{m}_q(a) + \mathrm{m}_q(b) \geq 2,$$

showing that C is not smooth at q. □

Remark. The preceding proposition is certainly not true for affine plane curves, since an affine plane curve can be the disjoint union of smooth curves, e.g.

$$C = \mathrm{V_a}(x) \cup \mathrm{V_a}(x-1) \subseteq \mathbb{A}^2.$$

There is a converse to the above proposition for degree 2 curves.

2.2.8 Proposition. *Let C be an irreducible projective plane curve of degree 2. Then C is smooth.*

Proof: Suppose instead that C is singular at p. Then if $C = \mathrm{V_p}(f)$, $\mathrm{m}_p(f) \geq 2$. Let q be another point on C and let L be the line joining p and q. Let h be a 1-form such that $L = \mathrm{V_p}(h)$. By Bézout's Theorem, p and q are the only two points of intersection of L and C. If L is not a component of C, then by Corollary 2.2.4,

$$\begin{aligned} 2 &= \deg(L) \cdot \deg(C) \\ &= \deg(h) \cdot \deg(f) \\ &= \mathrm{I}(p, L \cap C) + \mathrm{I}(q, L \cap C) \\ &\geq \mathrm{m}_p(L)\,\mathrm{m}_p(C) + \mathrm{m}_q(L)\,\mathrm{m}_p(C) \\ &\geq 2 + 1 \\ &= 3, \end{aligned}$$

which is absurd, so L must be a component of C, showing that C is reducible, a contradiction to our original assumption that C is singular at p. Therefore, C is smooth. □

Remark. In particular, this shows that any singular curve of degree 2 in $\mathbb{P}^2$ is the union of lines that intersect at the singular point, e.g.

$$C = \mathrm{V_p}(x^2 - y^2) = \mathrm{V_p}(x - y) \cup \mathrm{V_p}(x + y).$$