PMATH 764: Assignment 2

Due: Monday, 1 June, 2015.

1. Let k be an infinite field and $X \subset \mathbb{A}^n(k)$ be an irreducible algebraic set.
   (a) Prove that any non-empty Zariski open subset U of X is dense in X in the induced Zariski topology, i.e., $\overline{U} = X$.
   (b) Show that $U \cap V \neq \emptyset$ for any two non-empty open subsets $U, V \subset X$, and conclude that the (induced) Zariski topology fails to be Hausdorff on any irreducible algebraic set.
   (c) Prove that (a) and (b) are false if X is reducible.

2. Let E be the curve in $\mathbb{R}^2$ given in polar coordinates by $r = \theta$. Show that E is dense in $\mathbb{R}^2$ in the Zariski topology.

3. Let k be a infinite field (which may have characteristic 2).
   (a) Are the following ideals prime, radical, or closed in $k[x, y]$? Justify your answers.
      (i) $< x, y^2 - 1 >$
      (ii) $< x + y, xy >$
      (iii) $< x^3 - y^2 >$
   (b) Let $X_1, X_2 \subset \mathbb{A}^n(k)$ be algebraic sets. Prove the following:
      (i) $I(X_1 \cup X_2) = I(X_1) \cap (X_2)$;
      (ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$ if k is algebraically closed.

      Show that taking the radical in (ii) is, in general, necessary by finding algebraic sets $X_1, X_2 \subset \mathbb{A}^n(k)$ for which $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$; can you see geometrically what it means if we have inequality here?

4. (a) Let V, W be algebraic sets in $\mathbb{A}^n(k)$ with $V \subset W$. Show that each irreducible component of V is contained in some irreducible component of W.
   (b) If $X = X_1 \cup \cdots \cup X_k$ is the decomposition of an algebraic set X into irreducible components, show that $X_i \not\subset \bigcup_{j \neq i} X_j$.
   (c) Let $X = V(x^2 - yz, xz - x) \subset \mathbb{A}^3(k)$. Find the irreducible components of X. What are their prime ideals?

5. *(Optional)* Let R be a ring with identity and I be an ideal in R. Prove that there is a one-to-one correspondence between radical (resp. prime, resp. maximal) ideals of R containing I and radical (resp. prime, resp. maximal) ideals of R/I.

6. *Varieties.* An *affine variety* (or simply a *variety*) is defined to be an irreducible affine algebraic set. Let $X \subset \mathbb{A}^n(k)$ be a variety.
   (a) Verify that the induced Zariski topology on X coincides with the topology on X whose closed sets are the algebraic sets of $\mathbb{A}^n(k)$ contained in X.
   (b) A *subvariety* of X is defined as a variety $Y \subset \mathbb{A}^n(k)$ contained in X. Assume k is algebraically closed. Show that there is a one-to-one correspondence between algebraic subsets (resp. subvarieties, resp. points) of X and radical ideals (resp. prime ideals, resp. maximal ideals) of the quotient ring $\Gamma(X) := k[x_1, \ldots, x_n]/I(X)$.
   (c) Let Y be a subvariety of X and let $I_X(Y)$ be the ideal of $\Gamma(X)$ corresponding to Y. Prove that $\Gamma(Y)$ is isomorphic to $\Gamma(X)/I_X(Y)$.

7. *Affine schemes.* Let R be a commutative ring. The *spectrum of* R, denoted $\mathrm{Spec}(R)$, is defined as the set of all prime ideals in R.

   (a) Compute $\mathrm{Spec}(R)$ when R is

      (i) $\mathbb{Z}$;

      (ii) $\mathbb{Z}_6$;

      (iii) a field k;

      (iv) $\mathbb{R}[x]$;

      (v) $\mathbb{C}[x]$;

      (vi) $\mathbb{C}[x]/\langle x^2\rangle$.

   (b) For each $S \subset R$, let $V(S) := \{P \in \mathrm{Spec}(R) : S \subset P\}$. Show that the $V(S)$ are the closed sets of a topology on $\mathrm{Spec}(R)$ called the *Zariski topology.*

   An *affine scheme* consists of a pair of the form $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$, where $\mathrm{Spec}(R)$ is endowed with the Zariski topology and $\mathcal{O}_{\mathrm{Spec}(R)}$ is the *sheaf of regular functions on* $\mathrm{Spec}(R)$ (which we will define in subsequent assignments). For more details on affine schemes, see "The Geometry of Schemes" by D. Eisenbud and J. Harris, "The red book of varieties and schemes" by D. Mumford, or "Algebraic Geometry" by R. Hartshorne.

   (c) Show that, if R is a domain, the set $S = \{\langle 0\rangle\}$ consisting of the zero ideal is dense in $\mathrm{Spec}(R)$; in this case, the zero ideal is called the *generic point* of $\mathrm{Spec}(R)$.

   (d) For any field k, the spectrum $\mathbb{A}^n_k := \mathrm{Spec}(k[x_1, \ldots, x_n])$ is called *affine* n*-space.* Determine which points of the *affine lines* $\mathbb{A}^1_{\mathbb{R}}$ and $\mathbb{A}^1_{\mathbb{C}}$ are closed in the Zariski topology. Moreover, if k is an infinite field, describe all the points of the *affine plane* $\mathbb{A}^2_k$; how do these compare to the points of the affine plane we defined in class?