The Typical Structure of Intersecting Families

Balogh, Das, **Delcourt**, Liu, and Sharifzadeh

University of Illinois, at Urbana-Champaign

University of Waterloo
November 24, 2015

1. Definitions

- k-uniform hypergraphs

$$\{1,2,3\}$$
$$\{1,4,5\}$$
$$\{1,6,7\}$$
$$\{2,4,6\}$$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is* ***intersecting*** *if*
$\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq 1$

- k-uniform hypergraphs

$$\{1, 2, 3\}$$
$$\{1, 4, 5\}$$
$$\{1, 6, 7\}$$
$$\{2, 4, 6\}$$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is* ***intersecting*** *if* $\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq 1$

- k-uniform hypergraphs

$\{1, \color{red}{2}, 3\}$
$\{1, 4, 5\}$
$\{1, 6, 7\}$
$\{\color{red}{2}, 4, 6\}$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is **intersecting** if*
$\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq 1$

- k-uniform hypergraphs

$$\{1,2,3\}$$
$$\{1,4,5\}$$
$$\{1,6,7\}$$
$$\{2,4,6\}$$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is* ***intersecting*** *if*
$$\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq 1$$

- k-uniform hypergraphs

$$\{1,2,3\}$$
$$\{1,4,5\}$$
$$\{1,\textcolor{red}{6},7\}$$
$$\{2,4,\textcolor{red}{6}\}$$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is* ***intersecting*** *if*
$\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq 1$

- *k*-uniform hypergraphs

$\{1,2,3\}$ $\{1,3,7\}$
$\{1,2,4\}$ $\{1,4,5\}$
$\{1,2,5\}$ $\{1,4,6\}$
$\{1,2,6\}$ $\{1,4,7\}$
$\{1,2,7\}$ $\{1,5,6\}$
$\{1,3,4\}$ $\{1,5,7\}$
$\{1,3,5\}$ $\{1,6,7\}$
$\{1,3,6\}$

Definition

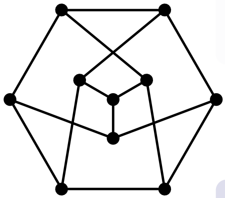
*intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is **trivial** if all edges share some vertex*

- k-uniform hypergraphs

$\{1, 2, 3\}$ $\{1, 3, 7\}$
$\{1, 2, 4\}$ $\{1, 4, 5\}$
$\{1, 2, 5\}$ $\{1, 4, 6\}$
$\{1, 2, 6\}$ $\{1, 4, 7\}$
$\{1, 2, 7\}$ $\{1, 5, 6\}$
$\{1, 3, 4\}$ $\{1, 5, 7\}$
$\{1, 3, 5\}$ $\{1, 6, 7\}$
$\{1, 3, 6\}$

Definition

intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is ***trivial***
if all edges share some vertex



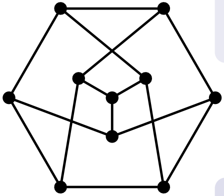
Definition

Kneser graph KG(n, k) on $\binom{[n]}{k}$ *has an edge if and only if the corresponding k-sets are disjoint*

independent sets in KG(n, k) correspond to intersecting families from $\binom{[n]}{k}$

Observation (Lovász 1979)

The minimum eigenvalue of KG(n, k) is $-\binom{n-k-1}{k-1}$.



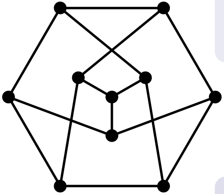
Definition

Kneser graph KG(n, k) on $\binom{[n]}{k}$ *has an edge if and only if the corresponding k-sets are disjoint*

independent sets in KG(n, k) correspond to intersecting families from $\binom{[n]}{k}$

Observation (Lovász 1979)

The minimum eigenvalue of KG(n, k) is $-\binom{n-k-1}{k-1}$.



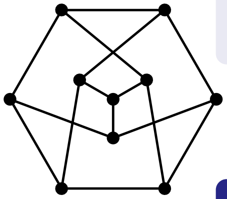
Definition

Kneser graph KG(n, k) on $\binom{[n]}{k}$ *has an edge if and only if the corresponding k-sets are disjoint*

independent sets in KG(n, k) correspond to intersecting families from $\binom{[n]}{k}$

Observation (Lovász 1979)

The minimum eigenvalue of KG(n, k) is $-\binom{n-k-1}{k-1}$.



Definition

Kneser graph KG(n, k) on $\binom{[n]}{k}$ *has an edge if and only if the corresponding k-sets are disjoint*

independent sets in KG(n, k) correspond to intersecting families from $\binom{[n]}{k}$

Observation (Lovász 1979)

The minimum eigenvalue of KG(n, k) is $-\binom{n-k-1}{k-1}$.

2. Stability Results

Maximal intersecting families

- k-uniform hypergraphs

$\{1,2,3\}$ $\{1,3,7\}$
$\{1,2,4\}$ $\{1,4,5\}$
$\{1,2,5\}$ $\{1,4,6\}$
$\{1,2,6\}$ $\{1,4,7\}$
$\{1,2,7\}$ $\{1,5,6\}$
$\{1,3,4\}$ $\{1,5,7\}$
$\{1,3,5\}$ $\{1,6,7\}$
$\{1,3,6\}$

Definition

*intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is **maximal** if it is not contained in a larger intersecting family*

Maximal intersecting families

- k-uniform hypergraphs

$\{1,2,3\}$ $\{1,3,7\}$
$\{1,2,4\}$ $\{1,4,5\}$
$\{1,2,5\}$ $\{1,4,6\}$
$\{1,2,6\}$ $\{1,4,7\}$
$\{1,2,7\}$ $\{1,5,6\}$
$\{1,3,4\}$ $\{1,5,7\}$
$\{1,3,5\}$ $\{1,6,7\}$
$\{1,3,6\}$

Definition

*intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is **maximal** if it is not contained in a larger intersecting family*

How large can an intersecting family be?

- k-uniform hypergraphs

$\{1,2,3\}$ $\{1,3,7\}$
$\{1,2,4\}$ $\{1,4,5\}$
$\{1,2,5\}$ $\{1,4,6\}$
$\{1,2,6\}$ $\{1,4,7\}$
$\{1,2,7\}$ $\{1,5,6\}$
$\{1,3,4\}$ $\{1,5,7\}$
$\{1,3,5\}$ $\{1,6,7\}$
$\{1,3,6\}$

for fixed $i \in [n]$,

$$\left|\left\{F \in \binom{[n]}{k} : i \in F\right\}\right| = \binom{n-1}{k-1}$$

$$\binom{7-1}{3-1} = \binom{6}{2} = 15$$

How large can an intersecting family be?

- k-uniform hypergraphs

$\{1,2,3\}$ $\{1,3,7\}$
$\{1,2,4\}$ $\{1,4,5\}$
$\{1,2,5\}$ $\{1,4,6\}$
$\{1,2,6\}$ $\{1,4,7\}$
$\{1,2,7\}$ $\{1,5,6\}$
$\{1,3,4\}$ $\{1,5,7\}$
$\{1,3,5\}$ $\{1,6,7\}$
$\{1,3,6\}$

for fixed $i \in [n]$,

$$\left|\left\{F \in \binom{[n]}{k} : i \in F\right\}\right| = \binom{n-1}{k-1}$$

$$\binom{7-1}{3-1} = \binom{6}{2} = 15$$

How large can an intersecting family be?

Theorem (Erdős-Ko-Rado 1961)

If $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then

$$e(\mathcal{F}) \leq \binom{n-1}{k-1}.$$

For $n > 2k$ we have equality only if $\mathcal{F}$ is trivial.

How large can an intersecting family be?

Theorem (Erdős-Ko-Rado Restated)

For $n \geq 2k$, the independence number of $KG(n,k)$ is less than or equal to $\binom{n-1}{k-1}$.

Theorem (Hoffman's bound)

Let $G = (V, E)$ be a d-regular graph and λ be the smallest eigenvalue. If $I \subseteq V$ is an independent set, then

$$|I| \leq |V| \frac{-\lambda}{d - \lambda}.$$

How large can an intersecting family be?

Theorem (Erdős-Ko-Rado Restated)

For $n \geq 2k$, the independence number of $KG(n,k)$ is less than or equal to $\binom{n-1}{k-1}$.

Theorem (Hoffman's bound)

Let $G = (V,E)$ be a d-regular graph and λ be the smallest eigenvalue. If $I \subseteq V$ is an independent set, then

$$|I| \leq |V| \frac{-\lambda}{d-\lambda}.$$

How large can an intersecting family be?

Theorem (Erdős-Ko-Rado Restated)

For $n \geq 2k$, the independence number of $KG(n,k)$ is less than or equal to $\binom{n-1}{k-1}$.

Proof.

Note that $KG(n,k)$ is a regular graph on $\binom{n}{k}$ vertices. Each vertex has degree $d = \binom{n-k}{k}$ (need $n \geq 2k$). The minimum eigenvalue of $KG(n,k)$ is $-\binom{n-k-1}{k-1}$. Using Hoffman's bound,

$$\alpha(KG(n,k)) \leq \binom{n}{k}\frac{-\lambda}{d-\lambda} = \binom{n}{k}\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.$$

$\square$

How large can an intersecting family be?

Theorem (Erdős-Ko-Rado Restated)

For $n \geq 2k$, the independence number of $KG(n, k)$ is less than or equal to $\binom{n-1}{k-1}$.

Proof.

Note that $KG(n, k)$ is a regular graph on $\binom{n}{k}$ vertices. Each vertex has degree $d = \binom{n-k}{k}$ (need $n \geq 2k$). The minimum eigenvalue of $KG(n, k)$ is $-\binom{n-k-1}{k-1}$. Using Hoffman's bound,

$$\alpha(KG(n, k)) \leq \binom{n}{k}\frac{-\lambda}{d-\lambda} = \binom{n}{k}\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.$$

$\square$

How large can an intersecting family be?

Theorem (Erdős-Ko-Rado Restated)

For $n \geq 2k$, the independence number of $KG(n,k)$ is less than or equal to $\binom{n-1}{k-1}$.

Proof.

Note that $KG(n,k)$ is a regular graph on $\binom{n}{k}$ vertices. Each vertex has degree $d = \binom{n-k}{k}$ (need $n \geq 2k$). The minimum eigenvalue of $KG(n,k)$ is $-\binom{n-k-1}{k-1}$. Using Hoffman's bound,

$$\alpha(KG(n,k)) \leq \binom{n}{k}\frac{-\lambda}{d-\lambda} = \binom{n}{k}\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.$$

□

How large can an intersecting family be?

Theorem (Erdős-Ko-Rado Restated)

For $n \geq 2k$, the independence number of $KG(n,k)$ is less than or equal to $\binom{n-1}{k-1}$.

Proof.

Note that $KG(n,k)$ is a regular graph on $\binom{n}{k}$ vertices. Each vertex has degree $d = \binom{n-k}{k}$ (need $n \geq 2k$). The minimum eigenvalue of $KG(n,k)$ is $-\binom{n-k-1}{k-1}$. Using Hoffman's bound,

$$\alpha(KG(n,k)) \leq \binom{n}{k}\frac{-\lambda}{d-\lambda} = \binom{n}{k}\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.$$

How large can an intersecting family be?

Theorem (Erdős-Ko-Rado Restated)

For $n \geq 2k$, the independence number of $KG(n,k)$ is less than or equal to $\binom{n-1}{k-1}$.

Proof.

Note that $KG(n,k)$ is a regular graph on $\binom{n}{k}$ vertices. Each vertex has degree $d = \binom{n-k}{k}$ (need $n \geq 2k$). The minimum eigenvalue of $KG(n,k)$ is $-\binom{n-k-1}{k-1}$. Using Hoffman's bound,

$$\alpha(KG(n,k)) \leq \binom{n}{k}\frac{-\lambda}{d-\lambda} = \binom{n}{k}\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.$$

$\square$

What about non-trivial families?

Theorem (Hilton-Milner 1967)

For $n > 2k$, the largest non-trivial intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ have size

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

3. Typical Structure

What does a "typical" family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any subset of a trivial intersecting family is itself trivial.

there are at least 2^{N_0} trivial families

What does a “typical” family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any subset of a trivial intersecting family is itself trivial.

there are at least 2^{N_0} trivial families

What does a "typical" family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any subset of a trivial intersecting family is itself trivial.

there are at least 2^{N_0} trivial families

What does a "typical" family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any subset of a trivial intersecting family is itself trivial.

there are at least 2^{N_0} trivial families

What does a “typical” family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any subset of a trivial intersecting family is itself trivial.

there are at least 2^{N_0} trivial families

What does a “typical” family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any non-trivial intersecting family must be contained inside a maximal non-trivial intersecting family.

there are at most $M2^{N_1}$ non-trivial families

What does a “typical” family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any non-trivial intersecting family must be contained inside a maximal non-trivial intersecting family.

there are at most $M2^{N_1}$ non-trivial families

What does a "typical" family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any non-trivial intersecting family must be contained inside a maximal non-trivial intersecting family.

there are at most $M2^{N_1}$ non-trivial families

What does a “typical” family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

If

$$\frac{M2^{N_1}}{2^{N_0}} \to 0,$$

then trivial families are **typical**.

Bounding the number of **maximal** families is interesting.

What does a "typical" family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

If

$$\frac{M2^{N_1}}{2^{N_0}} \to 0,$$

then trivial families are **typical**.

Bounding the number of **maximal** families is interesting.

What does a “typical” family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

If

$$\frac{M2^{N_1}}{2^{N_0}} \to 0,$$

then trivial families are **typical**.

Bounding the number of **maximal** families is interesting.

Set-pairs inequality

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

Corollary (BDDLS 2015)

Almost every intersecting family is trivially intersecting.

Set-pairs inequality

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

Corollary (BDDLS 2015)

Almost every intersecting family is trivially intersecting.

Set-pairs inequality

We will use the skew-symmetric Bollobás set-pairs inequality

Theorem (Frankl 1982)

Let $A_1, \ldots, A_m$ be sets of size a and $B_1, \ldots, B_m$ be sets of size b such that

$$A_i \cap B_i = \emptyset \text{ and } A_i \cap B_j \neq \emptyset$$

for every $1 \leq i < j \leq m$. Then $m \leq \binom{a+b}{a}$.

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{H}$	$\mathcal{I}(\mathcal{H})$	
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,3,7\}$
$\{1,2,4\}$	$\{1,2,4\}$	$\{1,4,5\}$
$\{1,3,4\}$	$\{1,2,5\}$	$\{1,4,6\}$
$\{1,5,6\}$	$\{1,2,6\}$	$\{1,4,7\}$
$\{1,5,7\}$	$\{1,2,7\}$	$\{1,5,6\}$
$\{1,6,7\}$	$\{1,3,4\}$	$\{1,5,7\}$
	$\{1,3,5\}$	$\{1,6,7\}$
	$\{1,3,6\}$	

- $\mathcal{H} \subseteq \binom{[n]}{k}$
- $\mathcal{I}(\mathcal{H}) := \left\{ G \in \binom{[n]}{k} : \forall F \in \mathcal{H}, |G \cap F| \geq 1 \right\}$
- $\mathcal{H}$ intersecting iff $\mathcal{H} \subseteq \mathcal{I}(\mathcal{H})$
- $\mathcal{H}$ maximal intersecting iff $\mathcal{H} = \mathcal{I}(\mathcal{H})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{H}$	$\mathcal{I}(\mathcal{H})$	
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,3,7\}$
$\{1,2,4\}$	$\{1,2,4\}$	$\{1,4,5\}$
$\{1,3,4\}$	$\{1,2,5\}$	$\{1,4,6\}$
$\{1,5,6\}$	$\{1,2,6\}$	$\{1,4,7\}$
$\{1,5,7\}$	$\{1,2,7\}$	$\{1,5,6\}$
$\{1,6,7\}$	$\{1,3,4\}$	$\{1,5,7\}$
	$\{1,3,5\}$	$\{1,6,7\}$
	$\{1,3,6\}$	

- $\mathcal{H} \subseteq \binom{[n]}{k}$
- $\mathcal{I}(\mathcal{H}) :=$ $\left\{ G \in \binom{[n]}{k} : \forall F \in \mathcal{H}, |G \cap F| \geq 1 \right\}$
- $\mathcal{H}$ intersecting iff $\mathcal{H} \subseteq \mathcal{I}(\mathcal{H})$
- $\mathcal{H}$ maximal intersecting iff $\mathcal{H} = \mathcal{I}(\mathcal{H})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{H}$	$\mathcal{I}(\mathcal{H})$	
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,3,7\}$
$\{1,2,4\}$	$\{1,2,4\}$	$\{1,4,5\}$
$\{1,3,4\}$	$\{1,2,5\}$	$\{1,4,6\}$
$\{1,5,6\}$	$\{1,2,6\}$	$\{1,4,7\}$
$\{1,5,7\}$	$\{1,2,7\}$	$\{1,5,6\}$
$\{1,6,7\}$	$\{1,3,4\}$	$\{1,5,7\}$
	$\{1,3,5\}$	$\{1,6,7\}$
	$\{1,3,6\}$	

- $\mathcal{H} \subseteq \binom{[n]}{k}$
- $\mathcal{I}(\mathcal{H}) := \left\{ G \in \binom{[n]}{k} : \forall F \in \mathcal{H}, |G \cap F| \geq 1 \right\}$
- $\mathcal{H}$ intersecting iff $\mathcal{H} \subseteq \mathcal{I}(\mathcal{H})$
- $\mathcal{H}$ maximal intersecting iff $\mathcal{H} = \mathcal{I}(\mathcal{H})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{H}$	$\mathcal{I}(\mathcal{H})$	
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,3,7\}$
$\{1,2,4\}$	$\{1,2,4\}$	$\{1,4,5\}$
$\{1,3,4\}$	$\{1,2,5\}$	$\{1,4,6\}$
$\{1,5,6\}$	$\{1,2,6\}$	$\{1,4,7\}$
$\{1,5,7\}$	$\{1,2,7\}$	$\{1,5,6\}$
$\{1,6,7\}$	$\{1,3,4\}$	$\{1,5,7\}$
	$\{1,3,5\}$	$\{1,6,7\}$
	$\{1,3,6\}$	

- $\mathcal{H} \subseteq \binom{[n]}{k}$
- $\mathcal{I}(\mathcal{H}) := \left\{ G \in \binom{[n]}{k} : \forall F \in \mathcal{H}, |G \cap F| \geq 1 \right\}$
- $\mathcal{H}$ intersecting iff $\mathcal{H} \subseteq \mathcal{I}(\mathcal{H})$
- $\mathcal{H}$ maximal intersecting iff $\mathcal{H} = \mathcal{I}(\mathcal{H})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{H}$	$\mathcal{I}(\mathcal{H})$	
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,3,7\}$
$\{1,2,4\}$	$\{1,2,4\}$	$\{1,4,5\}$
$\{1,3,4\}$	$\{1,2,5\}$	$\{1,4,6\}$
$\{1,5,6\}$	$\{1,2,6\}$	$\{1,4,7\}$
$\{1,5,7\}$	$\{1,2,7\}$	$\{1,5,6\}$
$\{1,6,7\}$	$\{1,3,4\}$	$\{1,5,7\}$
	$\{1,3,5\}$	$\{1,6,7\}$
	$\{1,3,6\}$	

- $\mathcal{H} \subseteq \binom{[n]}{k}$
- $\mathcal{I}(\mathcal{H}) := \left\{ G \in \binom{[n]}{k} : \forall F \in \mathcal{H}, |G \cap F| \geq 1 \right\}$
- $\mathcal{H}$ intersecting iff $\mathcal{H} \subseteq \mathcal{I}(\mathcal{H})$
- $\mathcal{H}$ maximal intersecting iff $\mathcal{H} = \mathcal{I}(\mathcal{H})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is at most*

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F} = \mathcal{I}(\mathcal{G})$		$\mathcal{G}$
$\{1,2,3\}$	$\{1,3,7\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,4,5\}$	$\{1,2,4\}$
$\{1,2,5\}$	$\{1,4,6\}$	$\{1,3,4\}$
$\{1,2,6\}$	$\{1,4,7\}$	$\{1,5,6\}$
$\{1,2,7\}$	$\{1,5,6\}$	$\{1,5,7\}$
$\{1,3,4\}$	$\{1,5,7\}$	$\{1,6,7\}$
$\{1,3,5\}$	$\{1,6,7\}$	
$\{1,3,6\}$		

- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{G} \subseteq \mathcal{F}$ is a **generating set** if $\mathcal{F} = \mathcal{I}(\mathcal{G})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F} = \mathcal{I}(\mathcal{G})$ $\quad$ $\mathcal{G}$

$\mathcal{F}$		$\mathcal{G}$
$\{1,2,3\}$	$\{1,3,7\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,4,5\}$	$\{1,2,4\}$
$\{1,2,5\}$	$\{1,4,6\}$	$\{1,3,4\}$
$\{1,2,6\}$	$\{1,4,7\}$	$\{1,5,6\}$
$\{1,2,7\}$	$\{1,5,6\}$	$\{1,5,7\}$
$\{1,3,4\}$	$\{1,5,7\}$	$\{1,6,7\}$
$\{1,3,5\}$	$\{1,6,7\}$	
$\{1,3,6\}$		

- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{G} \subseteq \mathcal{F}$ is a **generating set** if $\mathcal{F} = \mathcal{I}(\mathcal{G})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F} = \mathcal{I}(\mathcal{G})$		$\mathcal{G}$
$\{1,2,3\}$	$\{1,3,7\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,4,5\}$	$\{1,2,4\}$
$\{1,2,5\}$	$\{1,4,6\}$	$\{1,3,4\}$
$\{1,2,6\}$	$\{1,4,7\}$	$\{1,5,6\}$
$\{1,2,7\}$	$\{1,5,6\}$	$\{1,5,7\}$
$\{1,3,4\}$	$\{1,5,7\}$	$\{1,6,7\}$
$\{1,3,5\}$	$\{1,6,7\}$	
$\{1,3,6\}$		

- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{G} \subseteq \mathcal{F}$ is a **generating set** if $\mathcal{F} = \mathcal{I}(\mathcal{G})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is at most*

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F} = \mathcal{I}(\mathcal{F}_0)$

$\{1,2,3\}$ $\{1,2,4\}$ $\{1,2,5\}$ $\{1,2,6\}$ $\{1,2,7\}$ $\{1,3,4\}$ $\{1,3,5\}$ $\{1,3,6\}$ $\{1,3,7\}$ $\{1,4,5\}$ $\{1,4,6\}$ $\{1,4,7\}$ $\{1,5,6\}$ $\{1,5,7\}$ $\{1,6,7\}$

$\mathcal{F}_0$

$\{1,2,3\}$ $\{1,2,4\}$ $\{1,3,4\}$ $\{1,5,6\}$ $\{1,5,7\}$ $\{1,6,7\}$

- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{F}_0 = \{F_1, \ldots, F_s\} \subseteq \mathcal{F}$ minimal generating set of $\mathcal{F}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is at most*

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F} = \mathcal{I}(\mathcal{F}_0)$

$\{1,2,3\}$ $\{1,3,7\}$
$\{1,2,4\}$ $\{1,4,5\}$
$\{1,2,5\}$ $\{1,4,6\}$
$\{1,2,6\}$ $\{1,4,7\}$
$\{1,2,7\}$ $\{1,5,6\}$
$\{1,3,4\}$ $\{1,5,7\}$
$\{1,3,5\}$ $\{1,6,7\}$
$\{1,3,6\}$

$\mathcal{F}_0$

$\{1,2,3\}$
$\{1,2,4\}$
$\{1,3,4\}$
$\{1,5,6\}$
$\{1,5,7\}$
$\{1,6,7\}$

- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{F}_0 = \{F_1, \ldots, F_s\} \subseteq \mathcal{F}$ minimal generating set of $\mathcal{F}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{I}(\mathcal{F}_0 \setminus \{F_1\})$	F_1
	$\{1,2,3\}$
$\{1,2,3\}\{1,3,7\}$	
$\{1,2,4\}\{1,4,5\}$	$\mathcal{F}_0 \setminus \{F_1\}$
$\{1,2,5\}\{1,4,6\}$	$\{1,2,4\}$
$\{1,2,6\}\{1,4,7\}$	$\{1,3,4\}$
$\{1,2,7\}\{1,5,6\}$	$\{1,5,6\}$
$\{1,3,4\}\{1,5,7\}$	$\{1,5,7\}$
$\{1,3,5\}\{1,6,7\}$	$\{1,6,7\}$
$\{1,3,6\}\{4,5,6\}$	
$\{4,5,6\}\{4,6,7\}$	

- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{F}_0 = \{F_1, \ldots, F_s\} \subseteq \mathcal{F}$ minimal generating set of $\mathcal{F}$
- by minimality $\forall i \in [s]$, $\mathcal{F} \subsetneq \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\})$
- $\forall i \, \exists G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{F}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{I}(\mathcal{F}_0 \setminus \{F_1\})$	F_1
$\{1,2,3\}\{1,3,7\}$	$\{1,2,3\}$
$\{1,2,4\}\{1,4,5\}$	
$\{1,2,5\}\{1,4,6\}$	$\mathcal{F}_0 \setminus \{F_1\}$
$\{1,2,6\}\{1,4,7\}$	$\{1,2,4\}$
$\{1,2,7\}\{1,5,6\}$	$\{1,3,4\}$
$\{1,3,4\}\{1,5,7\}$	$\{1,5,6\}$
$\{1,3,5\}\{1,6,7\}$	$\{1,5,7\}$
$\{1,3,6\}\{4,5,6\}$	$\{1,6,7\}$
$\{4,5,6\}\{4,6,7\}$	

- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{F}_0 = \{F_1, \ldots, F_s\} \subseteq \mathcal{F}$ minimal generating set of $\mathcal{F}$
- by minimality $\forall i \in [s]$, $\mathcal{F} \subsetneq \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\})$
- $\forall i \; \exists G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{F}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is at most*

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{I}(\mathcal{F}_0 \setminus \{F_1\})$		F_1
		$\{1,2,3\}$
$\{1,2,3\}$	$\{1,3,7\}$	
$\{1,2,4\}$	$\{1,4,5\}$	$\mathcal{F}_0 \setminus \{F_1\}$
$\{1,2,5\}$	$\{1,4,6\}$	$\{1,2,4\}$
$\{1,2,6\}$	$\{1,4,7\}$	$\{1,3,4\}$
$\{1,2,7\}$	$\{1,5,6\}$	$\{1,5,6\}$
$\{1,3,4\}$	$\{1,5,7\}$	$\{1,5,7\}$
$\{1,3,5\}$	$\{1,6,7\}$	$\{1,6,7\}$
$\{1,3,6\}$	$\{4,5,6\}$	
$\{4,5,6\}$	$\{4,6,7\}$	

- $\forall i\ \exists G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{F}$
- $G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\})$
  $F_j \in \mathcal{F}_0 \setminus \{F_i\}, \forall j \neq i$
  $\forall j \neq i, |G_i \cap F_j| \geq 1$
- $G_i \notin \mathcal{F} = \mathcal{I}(\mathcal{F}_0)$
  $F_i \in \mathcal{F}_0$
  $|G_i \cap F_i| < 1$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{I}(\mathcal{F}_0 \setminus \{F_1\})$	F_1
	$\{1,2,3\}$
$\{1,2,3\}\{1,3,7\}$	
$\{1,2,4\}\{1,4,5\}$	$\mathcal{F}_0 \setminus \{F_1\}$
$\{1,2,5\}\{1,4,6\}$	$\{1,2,4\}$
$\{1,2,6\}\{1,4,7\}$	$\{1,3,4\}$
$\{1,2,7\}\{1,5,6\}$	$\{1,5,6\}$
$\{1,3,4\}\{1,5,7\}$	$\{1,5,7\}$
$\{1,3,5\}\{1,6,7\}$	$\{1,6,7\}$
$\{1,3,6\}\{4,5,6\}$	
$\{4,5,6\}\{4,6,7\}$	

- $\forall i \, \exists G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{F}$
- $G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\})$
  $F_j \in \mathcal{F}_0 \setminus \{F_i\}, \forall j \neq i$
  $\forall j \neq i, |G_i \cap F_j| \geq 1$
- $G_i \notin \mathcal{F} = \mathcal{I}(\mathcal{F}_0)$
  $F_i \in \mathcal{F}_0$
  $|G_i \cap F_i| < 1$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is at most*

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{I}(\mathcal{F}_0 \setminus \{F_1\})$		F_1
		$\{1,2,3\}$
$\{1,2,3\}$	$\{1,3,7\}$	
$\{1,2,4\}$	$\{1,4,5\}$	$\mathcal{F}_0 \setminus \{F_1\}$
$\{1,2,5\}$	$\{1,4,6\}$	$\{1,2,4\}$
$\{1,2,6\}$	$\{1,4,7\}$	$\{1,3,4\}$
$\{1,2,7\}$	$\{1,5,6\}$	$\{1,5,6\}$
$\{1,3,4\}$	$\{1,5,7\}$	$\{1,5,7\}$
$\{1,3,5\}$	$\{1,6,7\}$	$\{1,6,7\}$
$\{1,3,6\}$	$\{4,5,6\}$	
$\{4,5,6\}$	$\{4,6,7\}$	

- $\forall i \, \exists G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{F}$
- $G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\})$
  $F_j \in \mathcal{F}_0 \setminus \{F_i\}, \forall j \neq i$
  $\forall j \neq i, |G_i \cap F_j| \geq 1$
- $G_i \notin \mathcal{F} = \mathcal{I}(\mathcal{F}_0)$
  $F_i \in \mathcal{F}_0$
  $|G_i \cap F_i| < 1$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is at most*

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \le \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$$\begin{aligned}
A_1 &= \{1,2,3\} & B_1 &= \{4,5,6\} \\
A_2 &= \{1,2,4\} & B_2 &= \{3,5,6\} \\
A_3 &= \{1,3,4\} & B_3 &= \{2,5,6\} \\
A_4 &= \{1,5,6\} & B_4 &= \{2,3,7\} \\
A_5 &= \{1,5,7\} & B_5 &= \{2,3,6\} \\
A_6 &= \{1,6,7\} & B_6 &= \{2,3,5\}
\end{aligned}$$

- For $1 \le i \le s$, $A_i = F_i$ and $B_i = G_i$
- Then $|\mathcal{F}_0| \le \binom{2k}{k}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$$\begin{aligned} A_1 &= \{1,2,3\} & B_1 &= \{4,5,6\} \\ A_2 &= \{1,2,4\} & B_2 &= \{3,5,6\} \\ A_3 &= \{1,3,4\} & B_3 &= \{2,5,6\} \\ A_4 &= \{1,5,6\} & B_4 &= \{2,3,7\} \\ A_5 &= \{1,5,7\} & B_5 &= \{2,3,6\} \\ A_6 &= \{1,6,7\} & B_6 &= \{2,3,5\} \end{aligned}$$

- For $1 \leq i \leq s$, $A_i = F_i$ and $B_i = G_i$
- Then $|\mathcal{F}_0| \leq \binom{2k}{k}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$A_1 = \{1,2,3\}$ $B_1 = \{4,5,6\}$
$A_2 = \{1,2,4\}$ $B_2 = \{3,5,6\}$
$A_3 = \{1,3,4\}$ $B_3 = \{2,5,6\}$
$A_4 = \{1,5,6\}$ $B_4 = \{2,3,7\}$
$A_5 = \{1,5,7\}$ $B_5 = \{2,3,6\}$
$A_6 = \{1,6,7\}$ $B_6 = \{2,3,5\}$

- For $1 \leq i \leq s$, $A_i = F_i$ and $B_i = G_i$
- Then $|\mathcal{F}_0| \leq \binom{2k}{k}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F}_0$	$\mathcal{F}_0'$
$\{1,2,3\}$	$\{1,3,5\}$
$\{1,2,4\}$	$\{1,3,7\}$
$\{1,3,4\}$	$\{1,5,7\}$
$\{1,5,6\}$	$\{1,2,4\}$
$\{1,5,7\}$	$\{1,2,6\}$
$\{1,6,7\}$	$\{1,4,6\}$

- $\mathcal{F}_0$ is not necessarily unique
- because $\mathcal{F} = \mathcal{I}(\mathcal{F}_0)$, $\mathcal{F} \mapsto \mathcal{F}_0$ is an injection
- $|\mathcal{F}_0| \leq \binom{2k}{k}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F}_0$	$\mathcal{F}_0'$
$\{1,2,3\}$	$\{1,3,5\}$
$\{1,2,4\}$	$\{1,3,7\}$
$\{1,3,4\}$	$\{1,5,7\}$
$\{1,5,6\}$	$\{1,2,4\}$
$\{1,5,7\}$	$\{1,2,6\}$
$\{1,6,7\}$	$\{1,4,6\}$

- $\mathcal{F}_0$ is not necessarily unique
- because $\mathcal{F} = \mathcal{I}(\mathcal{F}_0)$, $\mathcal{F} \mapsto \mathcal{F}_0$ is an injection
- $|\mathcal{F}_0| \leq \binom{2k}{k}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F}_0$	$\mathcal{F}_0'$
$\{1,2,3\}$	$\{1,3,5\}$
$\{1,2,4\}$	$\{1,3,7\}$
$\{1,3,4\}$	$\{1,5,7\}$
$\{1,5,6\}$	$\{1,2,4\}$
$\{1,5,7\}$	$\{1,2,6\}$
$\{1,6,7\}$	$\{1,4,6\}$

- $\mathcal{F}_0$ is not necessarily unique
- because $\mathcal{F} = \mathcal{I}(\mathcal{F}_0)$, $\mathcal{F} \mapsto \mathcal{F}_0$ is an injection
- $|\mathcal{F}_0| \leq \binom{2k}{k}$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F}_0$	$\mathcal{F}_0'$
$\{1,2,3\}$	$\{1,3,5\}$
$\{1,2,4\}$	$\{1,3,7\}$
$\{1,3,4\}$	$\{1,5,7\}$
$\{1,5,6\}$	$\{1,2,4\}$
$\{1,5,7\}$	$\{1,2,6\}$
$\{1,6,7\}$	$\{1,4,6\}$

- the number of maximal intersecting hypergraphs is bounded by the number of sets of at most $\binom{2k}{k}$ edges
- being more clever we can get $\frac{1}{2}\binom{2k}{k}$ instead of $\binom{2k}{k}$ edges

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F}_0$	$\mathcal{F}_0'$
$\{1,2,3\}$	$\{1,3,5\}$
$\{1,2,4\}$	$\{1,3,7\}$
$\{1,3,4\}$	$\{1,5,7\}$
$\{1,5,6\}$	$\{1,2,4\}$
$\{1,5,7\}$	$\{1,2,6\}$
$\{1,6,7\}$	$\{1,4,6\}$

- the number of maximal intersecting hypergraphs is bounded by the number of sets of at most $\binom{2k}{k}$ edges
- being more clever we can get $\frac{1}{2}\binom{2k}{k}$ instead of $\binom{2k}{k}$ edges

4. Other Settings

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\, \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- For example,
  1 2 3 4
  1 3 4 2
  2 3 1 4

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\, \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- For example,
  1 2 3 4
  1 3 4 2
  2 3 1 4

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\, \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- For example,

  1 2 3 4
  1 3 4 2
  2 3 1 4

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\, \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- For example,
  1 2 3 4
  1 3 4 2
  2 3 1 4

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\, \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- For example,
  1 2 3 4
  1 3 4 2
  2 3 1 4

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\, \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- $\mathcal{F} \subseteq S_n$ is **trivial** if $\exists\, i, j \in [n]$ such that $\pi(i) = j \quad \forall\, \pi \in \mathcal{F}$
- For example,

  1 2 3 4
  1 2 4 3
  1 3 2 4
  1 3 4 2
  1 4 2 3
  1 4 3 2

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\, \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- $\mathcal{F} \subseteq S_n$ is **trivial** if $\exists\, i, j \in [n]$ such that $\pi(i) = j \quad \forall\, \pi \in \mathcal{F}$
- For example,
  1 2 3 4
  1 2 4 3
  1 3 2 4
  1 3 4 2
  1 4 2 3
  1 4 3 2

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\, \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- $\mathcal{F} \subseteq S_n$ is **trivial** if $\exists\, i, j \in [n]$ such that $\pi(i) = j \quad \forall\, \pi \in \mathcal{F}$
- For example,
  1 2 3 4
  1 2 4 3
  1 3 2 4
  1 3 4 2
  1 4 2 3
  1 4 3 2

How large can an intersecting family be?

1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2

for fixed $i, j \in [n]$,
$$|\{\sigma \in S_n : \sigma(i) = j\}| = (n-1)!$$

$$(4-1)! = 3! = 6$$

Theorem (Frankl–Deza 1977)

If $\mathcal{F} \subseteq S_n$ is intersecting, then $|\mathcal{F}| \leq (n-1)!$.

Theorem (Cameron–Ku / Larose–Malvenuto 2003)

We have equality above only if $\mathcal{F} \subseteq S_n$ is trivial.

How large can an intersecting family be?

1 2 3 4
1 2 4 3
1 3 2 4
1 3 4 2
1 4 2 3
1 4 3 2

for fixed $i, j \in [n]$,
$$|\{\sigma \in S_n : \sigma(i) = j\}| = (n-1)!$$

$$(4-1)! = 3! = 6$$

Theorem (Frankl–Deza 1977)

If $\mathcal{F} \subseteq S_n$ is intersecting, then $|\mathcal{F}| \leq (n-1)!$.

Theorem (Cameron–Ku / Larose–Malvenuto 2003)

We have equality above only if $\mathcal{F} \subseteq S_n$ is trivial.

How large can a non-trivial, intersecting family be?

Theorem (Ellis 2008)

For n sufficiently large, the largest non-trivial intersecting $\mathcal{F} \subseteq S_n$ have size

$$\left(1 - \frac{1}{e} + o(1)\right)(n-1)!.$$

$\{\sigma \in S_n : \sigma(1) = 1, \sigma(j) = j \text{ for some } j > 2\} \cup \{(1\ 2)\}$

How large can a non-trivial, intersecting family be?

Theorem (Ellis 2008)

For n sufficiently large, the largest non-trivial intersecting $\mathcal{F} \subseteq S_n$ have size

$$\left(1 - \frac{1}{e} + o(1)\right)(n-1)!.$$

$\{\sigma \in S_n : \sigma(1) = 1, \sigma(j) = j \text{ for some } j > 2\} \cup \{(1\ 2)\}$

How large can a non-trivial, intersecting family be?

Theorem (Ellis 2008)

For n sufficiently large, the largest non-trivial intersecting $\mathcal{F} \subseteq S_n$ have size

$$\left(1 - \frac{1}{e} + o(1)\right)(n-1)!.$$

$$\{\sigma \in S_n : \sigma(1) = 1, \sigma(j) = j \text{ for some } j > 2\} \cup \{(1\ 2)\}$$

Theorem (BDDLS 2015)

1. *The number of intersecting families of permutations is*

$$(n^2 + o(1))2^{(n-1)!}.$$

2. *Almost every intersecting family of permutations is trivial.*

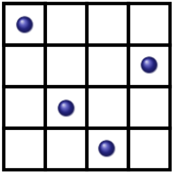
Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq S_n$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2n}{n}} \binom{n!}{i} < n^{n2^{2n-1}}.$$

$$\begin{array}{c} 1 \quad 3 \quad 4 \quad 2 \\ \{(1,1),(2,3),(3,4),(4,2)\} \end{array}$$



- to $\pi \in \mathcal{F}$, assign n-set of pairs $H_\pi = \{(1,\pi(1)),\dots,(n,\pi(n))\}$
- proof follows the same framework as before

Maximal intersecting families

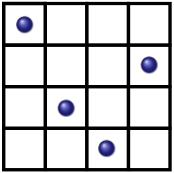
Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq S_n$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2n}{n}} \binom{n!}{i} < n^{n2^{2n-1}}.$$

1 3 4 2

$\{(1,1),(2,3),(3,4),(4,2)\}$



- to $\pi \in \mathcal{F}$, assign n-set of pairs $H_\pi = \{(1,\pi(1)),\dots,(n,\pi(n))\}$
- proof follows the same framework as before

t-intersecting families

$\{1, 2, 3\}$
$\{1, 2, 4\}$
$\{1, 2, 5\}$
$\{1, 2, 6\}$
$\{1, 2, 7\}$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is t-**intersecting** if*
$\forall F, G \in E(\mathcal{F}), \ |F \cap G| \geq t$

Definition

t-intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is **trivial***
if all edges share some t vertices

t-intersecting families

$\{1,2,3\}$
$\{1,2,4\}$
$\{1,2,5\}$
$\{1,2,6\}$
$\{1,2,7\}$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is t-**intersecting** if*
$\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq t$

Definition

*t-intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is **trivial***
if all edges share some t vertices

t-intersecting families

$\{1, 2, 3\}$
$\{1, 2, 4\}$
$\{1, 2, 5\}$
$\{1, 2, 6\}$
$\{1, 2, 7\}$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is t-**intersecting** if*
$\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq t$

Definition

*t-intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is **trivial***
if all edges share some t vertices

t-intersecting set-pairs inequality

Theorem (Füredi 1984)

Let $A_1, \ldots, A_m$ be sets of size a and $B_1, \ldots, B_m$ be sets of size b such that

$$|A_i \cap B_i| < t \text{ and } |A_i \cap B_j| \geq t$$

for every $1 \leq i < j \leq m$. Then $m \leq \binom{a+b-2t+2}{a-t+1}$.

Results

Theorem (BDDLS 2015)

1. *The number of t-intersecting families of $\binom{[n]}{k}$ is*

$$\left(\binom{n}{t} + o(1)\right) 2^{\binom{n-t}{k-t}}.$$

2. *Almost every t-intersecting family is trivial.*

Results

Theorem (BDDLS 2015)

1. *The number of t-intersecting families of S_n is*

$$\left(\binom{n}{t}^2 t! + o(1)\right) 2^{(n-t)!}.$$

2. *Almost every t-intersecting family of permutations is trivial.*

Method of graph containers

Proposition (BDDLS 2015)

The number of intersecting families of permutations is

$$2^{(1+o(1))(n-1)!}.$$

Theorem (Alon–Chung Expander Mixing Lemma (form in Alon–Balogh–Morris–Samotij 2014))

Let G be a D-regular graph on N vertices with second largest eigenvalue (in absolute value) λ. Then for all $S \subseteq V$,

$$e(G[S]) \geq \frac{D}{2N}|S|^2 + \frac{\lambda}{2N}|S|(N - |S|).$$

Method of graph containers

Proposition (BDDLS 2015)

The number of intersecting families of permutations is

$$2^{(1+o(1))(n-1)!}.$$

Consider the graph Γ with $V = S_n$ and edges non-intersecting pairs.

Theorem (Ellis 2011)

$\lambda = -(\frac{1}{e} + o(1))(n-1)!$

$N = n!$ and $D = (1 - \frac{1}{e} + o(1))N$.
Pick S with $|S| = (1 + o(1))(n-1)!$.
Because $G[S]$ spans 'many' edges then G does not have 'many' independent sets.

Method of graph containers

Proposition (BDDLS 2015)

The number of intersecting families of permutations is

$$2^{(1+o(1))(n-1)!}.$$

Consider the graph Γ with $V = S_n$ and edges non-intersecting pairs.

Theorem (Ellis 2011)

$\lambda = -(\frac{1}{e} + o(1))(n-1)!$

$N = n!$ and $D = (1 - \frac{1}{e} + o(1))N$.

Pick S with $|S| = (1 + o(1))(n-1)!$.

Because $G[S]$ spans 'many' edges then G does not have 'many' independent sets.

Method of graph containers

Proposition (BDDLS 2015)

The number of intersecting families of permutations is

$$2^{(1+o(1))(n-1)!}.$$

Consider the graph Γ with $V = S_n$ and edges non-intersecting pairs.

Theorem (Ellis 2011)

$\lambda = -(\frac{1}{e} + o(1))(n-1)!$

$N = n!$ and $D = (1 - \frac{1}{e} + o(1))N$.
Pick S with $|S| = (1 + o(1))(n-1)!$.
Because $G[S]$ spans 'many' edges then G does not have 'many' independent sets.

Method of graph containers

Proposition (BDDLS 2015)

The number of intersecting families of permutations is

$$2^{(1+o(1))(n-1)!}.$$

Consider the graph Γ with $V = S_n$ and edges non-intersecting pairs.

Theorem (Ellis 2011)

$\lambda = -(\frac{1}{e} + o(1))(n-1)!$

$N = n!$ and $D = (1 - \frac{1}{e} + o(1))N$.
Pick S with $|S| = (1 + o(1))(n-1)!$.
Because $G[S]$ spans 'many' edges then G does not have 'many' independent sets.

Thank you for listening!