Rainbow Copies of C_4 in Edge-Colored Hypercubes

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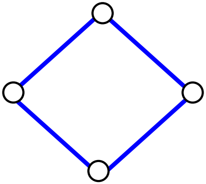
Definitions

Monochromatic Coloring

For a graph G, an edge coloring

$$\varphi : E(G) \to \{1, 2, \ldots\}$$

is called **monochromatic** if all edges receive the same color.



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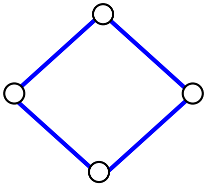

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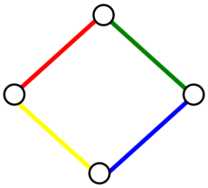

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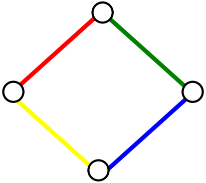
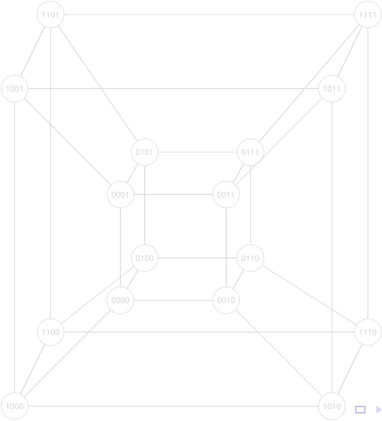


d-dimensional Hypercube

Let $\mathcal{Q}_d$ have vertices corresponding elements of $\{0, 1\}^d$ and put edges between elements of Hamming distance 1.



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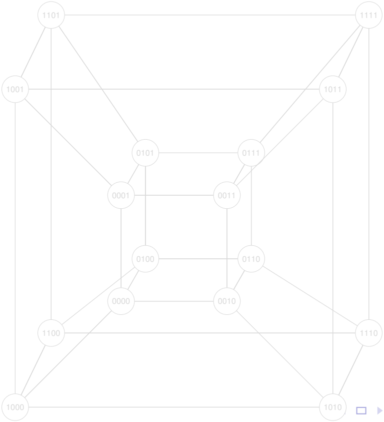

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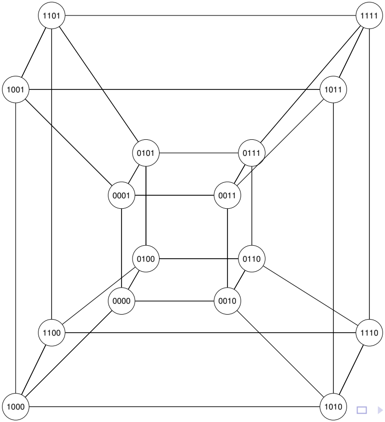

Motivation

Rainbow Variants

Many classical problems have rainbow variants.

In Classical Extremal Graph Theory
we have conditions on a graph to guarantee the existence of a set of subgraphs (e.g. Ramsey and Turán type problems).

In Rainbow-type Problems
we have conditions on a graph to guarantee the existence of a set of rainbow subgraphs.

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Rainbow Variants

For a graph H, let $\gcd(H)$ denote the largest integer which divides the degree of every vertex of H.

Yuster proved a variant of Wilson's Theorem.

Theorem (Yuster)

For every H there exists $N \in \mathbb{N}$ such that if $n > N$, $e(H) \mid \binom{n}{2}$, and $\gcd(H) \mid n-1$, then a properly edge-colored K_n has an H-decomposition with each copy of H rainbow.

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Rainbow Variants

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Erdős, Simonovits, and Sós looked at the maximum number of colors in an edge coloring of K_n with no rainbow copy of H (Anti-Ramsey problems).

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In particular Erdős, Simonovits, and Sós studied cycles.

Conjecture (Erdős, Simonovits, and Sós)

It is possible to color the edges of K_n with

$$n\left(\frac{k-2}{2}+\frac{1}{k-1}\right)+O(1)$$

colors such that there is no rainbow C_k.

- True for C_3 (Erdős, Simonovits, and Sós)
- True for C_4 (Alon)
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Edge-Colorings of Hypercubes

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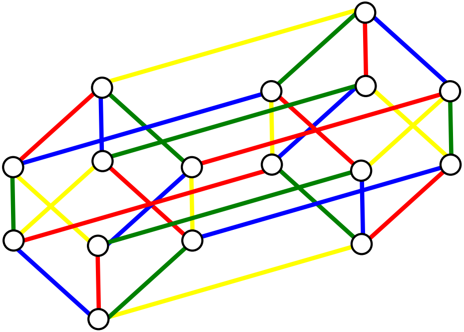

$d = 5$

Omitting a proof, Faudree, Gyárfás, Lesniak, and Schelp claim that there is no 5-edge-coloring of $\mathcal{Q}_5$ where every copy of C_4 is rainbow.

Using a computer, we find that the maximum number of rainbow copies of C_4 in a 5-edge-coloring of $\mathcal{Q}_5$ is 73 out of the

$$d(d-1)2^{d-3} = 80$$

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Generalization

We studied a generalization.

For any $k, d \in \mathbb{N}$ with $4 \leq k < d$ and $k \neq 5$, we find the maximum number of rainbow copies of C_4 contained in a k-edge-coloring of $\mathcal{Q}_d$.

The k-edge-colorings of $\mathcal{Q}_d$ with the maximum number of rainbow copies of C_4 also have the property that every non-rainbow C_4 is actually monochromatic.

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Note that when $k = d$, by Faudree, Gyárfás, Lesniak, and Schelp, there is an edge-coloring of $\mathcal{Q}_d$ using d colors where **every** C_4 is rainbow.

Thus, there at most

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$k < d$

Our main result is the following theorem.

Theorem

Fix integers k and d such that $4 \leq k < d$ and $k \neq 5$ and write

$$d = ka + b$$

*such that $a \in \mathbb{N}$ and $b \in \{0, 1, 2, \ldots, k-1\}$.
Then the maximum number of rainbow copies of C_4 in a k-edge-coloring of Q_d is*

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Upper bound

Assume that $\mathcal{Q}_d$ is edge-colored with colors

$$[k] = \{1, \ldots, k\}$$

such that the number of rainbow copies of C_4 is maximized.

A vertex in $\mathcal{Q}_d$, say v, has $\binom{d}{2}$ incident copies of C_4.

In the set of t_i edges of color $i \in [k]$ which are incident to v, none of the $\binom{t_i}{2}$ possible pairs can be in a rainbow copy of C_4.

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If the color classes are as equal as possible and

$$t_1 + \ldots + t_k = d = ka + b,$$

then there are at most

$$\binom{d}{2} - \sum_{i \in [k]} \binom{t_i}{2} \leq \binom{d}{2} - (k-b)\binom{a}{2} - b\binom{a+1}{2}$$

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Lower bound

We would like to use edge-coloring of $\mathcal{Q}_k$ to color edges of $\mathcal{Q}_d$.

Now we give a construction using a "blow-up technique".

Thinking of vertices of $\mathcal{Q}_d$ as elements of $\{0,1\}^d$, we want to partition each string into k "blocks" of consecutive binary digits of length either a or $a+1$.

We partition the first $(k-b)a$ binary digits into $(k-b)$ blocks of length a and the last $b(a+1)$ digits into b blocks of length $a+1$.

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We associate an element of $\{0, 1\}^k$ with each vertex of $\mathcal{Q}_d$ by computing the sum of the terms in each block modulo 2.

This process gives a map

$$h : V(\mathcal{Q}_d) \to V(\mathcal{Q}_k).$$

For example, consider $d = 10$ and $k = 3$:

$$\begin{array}{c|c|c} 111 & 011 & 1011 \\ 1 & 0 & 1 \end{array}$$

and

$$h(1110111011) = 101.$$

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We associate an element of $\{0, 1\}^k$ with each vertex of $\mathcal{Q}_d$ by computing the sum of the terms in each block modulo 2.

This process gives a map

$$h : V(\mathcal{Q}_d) \to V(\mathcal{Q}_k).$$

For example, consider $d = 10$ and $k = 3$:

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Furthermore, h preserves edges.

Edges of $\mathcal{Q}_d$ are pairs of vertices with Hamming distance 1.

If $u, v \in V(\mathcal{Q}_d)$ have Hamming distance 1, then $h(u)$ and $h(v)$ differ exactly in one block and have Hamming distance 1.

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Faudree, Gyárfás, Lesniak, and Schelp showed there is a k-edge-coloring of $\mathcal{Q}_k$, say φ, such that every C_4 is rainbow.

Color edges of $\mathcal{Q}_d$ with the color of their image under h in $\mathcal{Q}_k$, i.e. the color of an edge e in $\mathcal{Q}_d$ is $\varphi(h(e))$.

Using this coloring, each vertex in $\mathcal{Q}_d$ is incident to d edges, a edges of each of $k - b$ colors and $a + 1$ edges of each of the remaining b colors.

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We must check that for each vertex v in $\mathcal{Q}_d$, each pair of edges with different colors incident to v is actually in a rainbow C_4.

Note that among the four vertices in any C_4 the maximum Hamming distance is 2.

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If all the differences occur in the same block,
then the four edges of the C_4 are mapped to the same edge in $\mathcal{Q}_k$, and thus, the C_4 is monochromatic in $\mathcal{Q}_d$.

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Further Directions

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For $k = 5$, flag algebra methods did not improve the upper bound obtained from our main result.

We actually suspect that the upper bound might be the correct order of magnitude for large d.

Lower Bound

For a lower bound, our blow-up method can be applied to a 5-edge-coloring of Q_5 with 73 rainbow copies of C_4.

This, however, is very far from our upper bound.

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Edge-Coloring Subgraphs

Let G and H be graphs and $|E(H)| \geq q \in \mathbb{N}$.

Denote the minimum number of colors required to edge-color G such that the edges of every copy of H in G receive at least q colors by

$$f(G, H, q).$$

In this context, Faudree, Gyárfás, Lesniak, and Schelp show

$$f(\mathcal{Q}_d, C_4, |E(C_4)|) = f(\mathcal{Q}_d, C_4, 4) = d,$$

for integer $4 \leq d$ with $d \neq 5$.

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Mubayi and Stading generalized this result.

They proved that there are positive constants, say c_1 and c_2, depending only on k such that

$$c_1 d^{k/4} < f(\mathcal{Q}_d, C_k, k) < c_2 d^{k/4}$$

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Thank you for listening!