Definitions

High Girth and High Chromatic Number

Random Regular Graphs

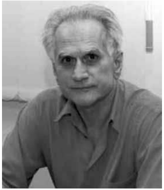
3-Flow Conjecture

Introduction to Extremal and Probabilistic Combinatorics

Michelle Delcourt

Graduate Student Colloquium
University of Illinois at Urbana-Champaign

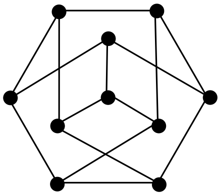
September 7, 2016



Definitions

Definition (Graph)

A graph is an ordered pair $G = (V, E)$ consisting of a vertex set V and set of edges E (2-element subsets of V).



Extremal Graph Theory

How much of something can you have, given a certain constraint?

Probabilistic Methods

Technique for proving the existence of combinatorial objects with specified properties.

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Extremal Graph Theory
How much of something can you have, given a certain constraint?

Probabilistic Methods
Technique for proving the existence of combinatorial objects with specified properties.

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Extremal Graph Theory
How much of something can you have, given a certain constraint?

Probabilistic Methods
Technique for proving the existence of combinatorial objects with specified properties.

Definitions

High Girth and High Chromatic Number

Random Regular Graphs

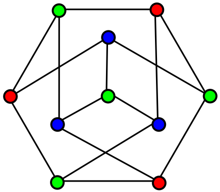
3-Flow Conjecture

Extremal Graph Theory
How much of something can you have, given a certain constraint?

Probabilistic Methods
Technique for proving the existence of combinatorial objects with specified properties.

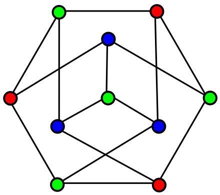
Definition (Proper Coloring)

A proper coloring of G is an assignment of labels (or colors) to vertices such that no edge connects two vertices with the same color.



Definition (k-coloring)

A proper coloring of G with k (or fewer colors) is a k-coloring.

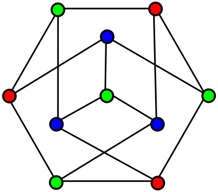


Definition (Chromatic Number)

The chromatic number of G is the smallest k such that there is a k-coloring of G.

Definition (k-coloring)

A proper coloring of G with k (or fewer colors) is a k-coloring.

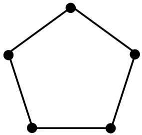


Definition (Chromatic Number)

The chromatic number of G is the smallest k such that there is a k-coloring of G.

Definition (Cycle of length k)

A cycle of length k consists of a closed walk (no repetitions of vertices or edges) through k vertices.



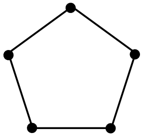
Definition (Girth)

The girth of a graph G is the length of a shortest cycle contained in G.

Observe: a triangle-free graph has girth ≥ 4.

Definition (Cycle of length k)

A cycle of length k consists of a closed walk (no repetitions of vertices or edges) through k vertices.



Definition (Girth)

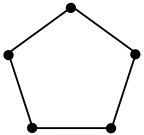
The girth of a graph G is the length of a shortest cycle contained in G.

Observe: a triangle-free graph has girth ≥ 4.

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Definition (Cycle of length k)

A cycle of length k consists of a closed walk (no repetitions of vertices or edges) through k vertices.

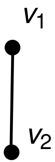


Definition (Girth)

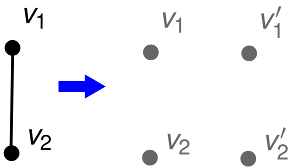
The girth of a graph G is the length of a shortest cycle contained in G.

Observe: a triangle-free graph has girth ≥ 4.

In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.

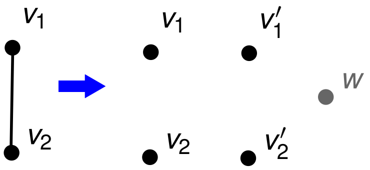


In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.

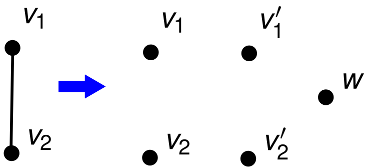


Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

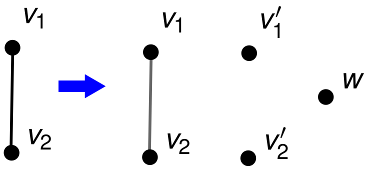
In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



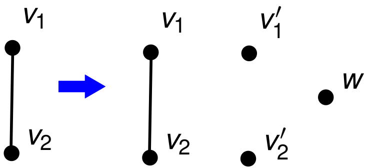
In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



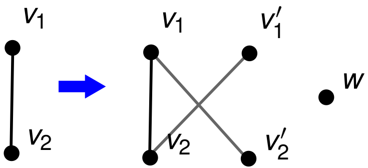
Definitions

High Girth and High Chromatic Number

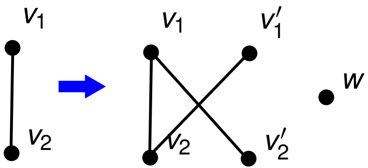
Random Regular Graphs

3-Flow Conjecture

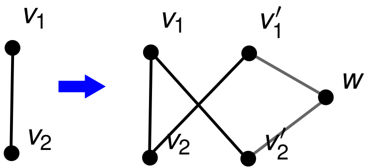
In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



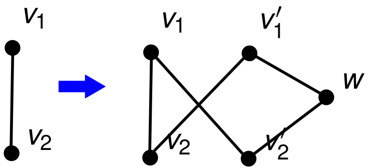
In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



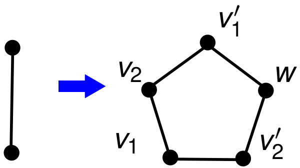
In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.

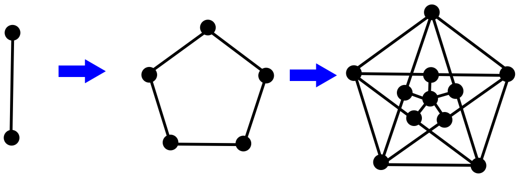


In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

In 1955, Mycielski constructed triangle-free graphs with arbitrarily high chromatic number.



Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

High Girth
and
High Chromatic Number

What about for higher girth?

Can we find graphs with high girth and arbitrarily high chromatic number?

Yes, breakthrough using probabilistic combinatorics

What about for higher girth?

Can we find graphs with high girth and arbitrarily high chromatic number?

Yes, breakthrough using probabilistic combinatorics

What about for higher girth?

Can we find graphs with high girth and arbitrarily high chromatic number?

Yes, breakthrough using probabilistic combinatorics

GRAPH THEORY AND PROBABILITY

P. ERDŐS

A well-known theorem of Ramsay (**8; 9**) states that to every n there exists a smallest integer $g(n)$ so that every graph of $g(n)$ vertices contains either a set of n independent points or a complete graph of order n, but there exists a graph of $g(n) - 1$ vertices which does not contain a complete subgraph of n vertices and also does not contain a set of n independent points. (A graph is called complete if every two of its vertices are connected by an edge; a set of points is called independent if no two of its points are connected by an edge.) The determination of $g(n)$ seems a very difficult problem; the best inequalities for $g(n)$ are (**3**)

$$(1) \qquad 2^{\frac{1}{2}n} < g(n) \leqslant \binom{2n-2}{n-1}.$$

It is not even known that $g(n)^{1/n}$ tends to a limit. The lower bound in (1) has been obtained by combinatorial and probabilistic arguments without an explicit construction.

In our paper (**5**) with Szekeres $f(k, l)$ is defined as the least integer so that every graph having $f(k, l)$ vertices contains either a complete graph of order k or a set of l independent points ($f(k, k) = g(k)$). Szekeres proved

$$(2) \qquad f(k, l) \leqslant \binom{k+l-2}{k-1}.$$

Thus for

$$k = 3, f(3, l) \leqslant \binom{l+1}{2}.$$

I recently proved by an explicit construction that $f(3, l) > l^{1+c_1}$ (**4**). By probabilistic arguments I can prove that for $k > 3$

$$(3) \qquad f(k, l) > l\binom{k+l-2}{k-1}^{c_2},$$

which shows that (2) is not very far from being best possible.

Define now $h(k, l)$ as the least integer so that every graph of $h(k, l)$ vertices contains either a closed circuit of k or fewer lines, or that the graph contains a set of l independent points. Clearly $h(3, l) = f(3, l)$.

By probabilistic arguments we are going to prove that for fixed k and sufficiently large l

$$(4) \qquad h(k, l) > l^{1+1/2k}.$$

Further we shall prove that

Received December 13, 1957.

GRAPH THEORY AND PROBABILITY

P. ERDŐS

A well-known theorem of Ramsay **(8; 9)** states that to every n there exists a smallest integer $g(n)$ so that every graph of $g(n)$ vertices contains either a set of n independent points or a complete graph of order n, but there exists a graph of $g(n) - 1$ vertices which does not contain a complete subgraph of n vertices and also does not contain a set of n independent points. (A graph is called complete if every two of its vertices are connected by an edge; a set of points is called independent if no two of its points are connected by an edge.) The determination of $g(n)$ seems a very difficult problem; the best inequalities for $g(n)$ are **(3)**

$$2^{\frac{1}{2}n} < g(n) \leqslant \binom{2n-2}{n-1}. \tag{1}$$

It is not even known that $g(n)^{1/n}$ tends to a limit. The lower bound in (1) has been obtained by combinatorial and probabilistic arguments without an explicit construction.

In our paper **(5)** with Szekeres $f(k, l)$ is defined as the least integer so that every graph having $f(k, l)$ vertices contains either a complete graph of order k or a set of l independent points ($f(k, k) = g(k)$). Szekeres proved

$$f(k, l) \leqslant \binom{k+l-2}{k-1}. \tag{2}$$

Thus for

$$k = 3, f(3, l) \leqslant \binom{l+1}{2}.$$

I recently proved by an explicit construction that $f(3, l) > l^{1+c_1}$ **(4)**. By probabilistic arguments I can prove that for $k > 3$

$$f(k, l) > l\binom{k+l-2}{k-1}^{c_2}, \tag{3}$$

which shows that (2) is not very far from being best possible.

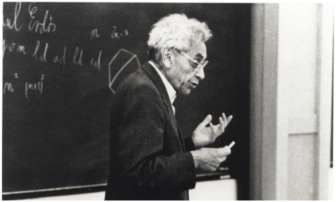
Define now $h(k, l)$ as the least integer so that every graph of $h(k, l)$ vertices contains either a closed circuit of k or fewer lines, or that the graph contains a set of l independent points. Clearly $h(3, l) = f(3, l)$.

By probabilistic arguments we are going to prove that for fixed k and sufficiently large l

$$h(k, l) > l^{1+1/2k}. \tag{4}$$

Further we shall prove that

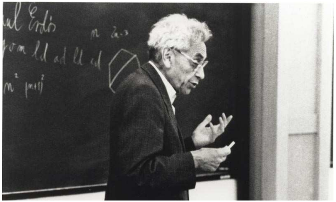
Received December 13, 1957.



Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

"It is not enough to be in the right place at the right time.

You should also have an open mind at the right time."
–Paul Erdős



GRAPH THEORY AND PROBABILITY

P. ERDŐS

A well-known theorem of Ramsay **(8; 9)** states that to every n there exists a smallest integer $g(n)$ so that every graph of $g(n)$ vertices contains either a set of n independent points or a complete graph of order n, but there exists a graph of $g(n) - 1$ vertices which does not contain a complete subgraph of n vertices and also does not contain a set of n independent points. (A graph is called complete if every two of its vertices are connected by an edge; a set of points is called independent if no two of its points are connected by an edge.) The determination of $g(n)$ seems a very difficult problem; the best inequalities for $g(n)$ are **(3)**

$$2^{\frac{1}{2}n} < g(n) \leqslant \binom{2n-2}{n-1}. \tag{1}$$

It is not even known that $g(n)^{1/n}$ tends to a limit. The lower bound in (1) has been obtained by combinatorial and probabilistic arguments without an explicit construction.

In our paper **(5)** with Szekeres $f(k, l)$ is defined as the least integer so that every graph having $f(k, l)$ vertices contains either a complete graph of order k or a set of l independent points $(f(k, k) = g(k))$. Szekeres proved

$$f(k, l) \leqslant \binom{k+l-2}{k-1}. \tag{2}$$

Thus for

$$k = 3, f(3, l) \leqslant \binom{l+1}{2}.$$

I recently proved by an explicit construction that $f(3, l) > l^{1+c_1}$ **(4)**. By probabilistic arguments I can prove that for $k > 3$

$$f(k, l) > l\binom{k+l-2}{k-1}^{c_2}. \tag{3}$$

which shows that (2) is not very far from being best possible.

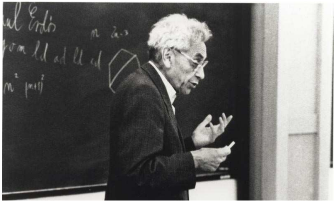
Define now $h(k, l)$ as the least integer so that every graph of $h(k, l)$ vertices contains either a closed circuit of k or fewer lines, or that the graph contains a set of l independent points. Clearly $h(3, l) = f(3, l)$.

By probabilistic arguments we are going to prove that for fixed k and sufficiently large l

$$h(k, l) > l^{1+1/2k}. \tag{4}$$

Further we shall prove that

Received December 13, 1957.



Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

Idea: use random graphs

How do we generate random graphs on n vertices?

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

Idea: use random graphs

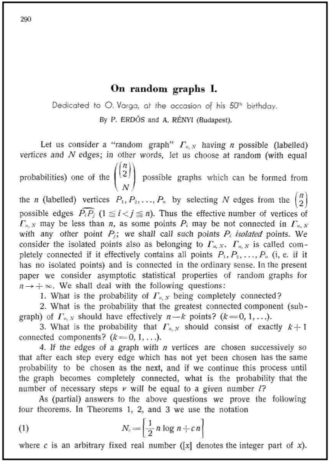
How do we generate random graphs on n vertices?

Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

Idea: use random graphs

How do we generate random graphs on n vertices?



290

On random graphs I.

Dedicated to O. Varga, at the occasion of his 50th birthday.

By P. ERDŐS and A. RÉNYI (Budapest).

Let us consider a "random graph" $\Gamma_{n,N}$ having n possible (labelled) vertices and N edges; in other words, let us choose at random (with equal probabilities) one of the $\binom{\binom{n}{2}}{N}$ possible graphs which can be formed from the n (labelled) vertices $P_1, P_2, \ldots, P_n$ by selecting N edges from the $\binom{n}{2}$ possible edges $\overline{P_i P_j}$ $(1 \leq i < j \leq n)$. Thus the effective number of vertices of $\Gamma_{n,N}$ may be less than n, as some points P_i may be not connected in $\Gamma_{n,N}$ with any other point P_j; we shall call such points P_i *isolated* points. We consider the isolated points also as belonging to $\Gamma_{n,N}$. $\Gamma_{n,N}$ is called completely connected if it effectively contains all points $P_1, P_2, \ldots, P_n$ (i. e. if it has no isolated points) and is connected in the ordinary sense. In the present paper we consider asymptotic statistical properties of random graphs for $n \to +\infty$. We shall deal with the following questions:

1. What is the probability of $\Gamma_{n,N}$ being completely connected?

2. What is the probability that the greatest connected component (sub-graph) of $\Gamma_{n,N}$ should have effectively $n-k$ points? $(k = 0, 1, \ldots)$.

3. What is the probability that $\Gamma_{n,N}$ should consist of exactly $k+1$ connected components? $(k = 0, 1, \ldots)$.

4. If the edges of a graph with n vertices are chosen successively so that after each step every edge which has not yet been chosen has the same probability to be chosen as the next, and if we continue this process until the graph becomes completely connected, what is the probability that the number of necessary steps ν will be equal to a given number l?

As (partial) answers to the above questions we prove the following four theorems. In Theorems 1, 2, and 3 we use the notation

$$(1) \qquad N_c = \left[\frac{1}{2} n \log n + cn\right]$$

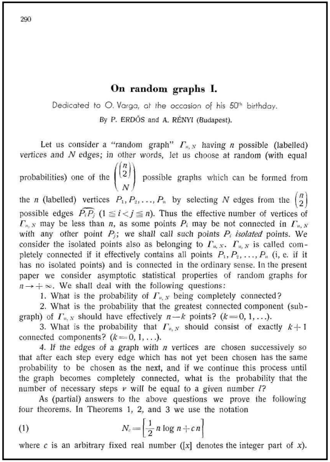
where c is an arbitrary fixed real number ($[x]$ denotes the integer part of x).

"A mathematician is a device for turning coffee into theorems."
–Paul Erdős

"A mathematician is a device for turning coffee into theorems."
–Alfréd Rényi

"If I feel unhappy,
I do mathematics
to become happy.

If I am happy,
I do mathematics
to keep happy."
–Alfréd Rényi



On random graphs I.

Dedicated to O. Varga, at the occasion of his 50th birthday.

By P. ERDŐS and A. RÉNYI (Budapest).

Let us consider a "random graph" $\Gamma_{n,N}$ having n possible (labelled) vertices and N edges; in other words, let us choose at random (with equal probabilities) one of the $\binom{\binom{n}{2}}{N}$ possible graphs which can be formed from the n (labelled) vertices $P_1, P_2, \ldots, P_n$ by selecting N edges from the $\binom{n}{2}$ possible edges $\overline{P_iP_j}$ $(1 \leq i < j \leq n)$. Thus the effective number of vertices of $\Gamma_{n,N}$ may be less than n, as some points P_i may be not connected in $\Gamma_{n,N}$ with any other point P_j; we shall call such points P_i *isolated* points. We consider the isolated points also as belonging to $\Gamma_{n,N}$. $\Gamma_{n,N}$ is called completely connected if it effectively contains all points $P_1, P_2, \ldots, P_n$ (i. e. if it has no isolated points) and is connected in the ordinary sense. In the present paper we consider asymptotic statistical properties of random graphs for $n \to +\infty$. We shall deal with the following questions:

1. What is the probability of $\Gamma_{n,N}$ being completely connected?

2. What is the probability that the greatest connected component (sub-graph) of $\Gamma_{n,N}$ should have effectively $n-k$ points? $(k = 0, 1, \ldots)$.

3. What is the probability that $\Gamma_{n,N}$ should consist of exactly $k+1$ connected components? $(k = 0, 1, \ldots)$.

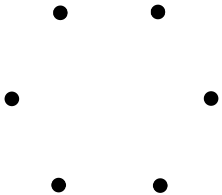
4. If the edges of a graph with n vertices are chosen successively so that after each step every edge which has not yet been chosen has the same probability to be chosen as the next, and if we continue this process until the graph becomes completely connected, what is the probability that the number of necessary steps ν will be equal to a given number l?

As (partial) answers to the above questions we prove the following four theorems. In Theorems 1, 2, and 3 we use the notation

$$N_c = \left[\frac{1}{2} n \log n + cn\right] \tag{1}$$

where c is an arbitrary fixed real number ($[x]$ denotes the integer part of x).

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

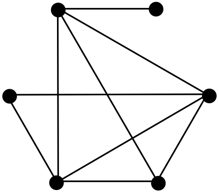


$n = 6$

Model (Erdős and Rényi 1959)

$G(n, p)$ model (Erdős–Rényi model)

1. *Begin with n vertices.*
2.



$n = 6, p = \frac{1}{2}$

Model (Erdős and Rényi 1959)

$G(n, p)$ model (Erdős–Rényi model)

1. *Begin with n vertices.*
2. *Include each edge independently with probability p.*

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

Idea:

- for n large and p carefully chosen, $G_{n,p}$ has "few" short cycles (at least half the time)
- for n large $G_{n,p}$ has high chromatic number (at least half the time)
- combining these and deleting some problem vertices we get graphs with high chromatic number and no short cycles

Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

Idea:

- for n large and p carefully chosen, $G_{n,p}$ has "few" short cycles (at least half the time)
- for n large $G_{n,p}$ has high chromatic number (at least half the time)
- combining these and deleting some problem vertices we get graphs with high chromatic number and no short cycles

Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

Idea:

- for n large and p carefully chosen, $G_{n,p}$ has "few" short cycles (at least half the time)
- for n large $G_{n,p}$ has high chromatic number (at least half the time)
- combining these and deleting some problem vertices we get graphs with high chromatic number and no short cycles

Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

Idea:

- for n large and p carefully chosen, $G_{n,p}$ has “few” short cycles (at least half the time)
- for n large $G_{n,p}$ has high chromatic number (at least half the time)
- combining these and deleting some problem vertices we get graphs with high chromatic number and no short cycles

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

this is an existence proof, not a construction

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Theorem (Erdős 1959)

For any integers ℓ and k, there is a graph of girth $> \ell$ and chromatic number $> k$.

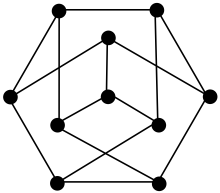
this is an existence proof, not a construction

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Random Regular Graphs

Definition (Regular)

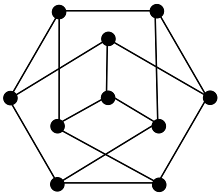
G is regular if all vertices have the same degree.



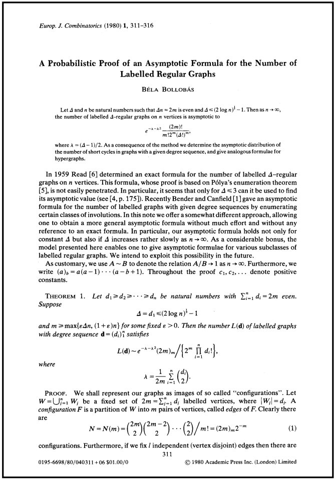
How do we generate random d-regular graphs on n vertices?

Definition (Regular)

G is regular if all vertices have the same degree.



How do we generate random d-regular graphs on n vertices?



Europ. J. Combinatorics (1980) **1**, 311–316

A Probabilistic Proof of an Asymptotic Formula for the Number of Labelled Regular Graphs

BÉLA BOLLOBÁS

Let Δ and n be natural numbers such that $\Delta n = 2m$ is even and $\Delta \leqslant (2 \log n)^{\frac{1}{2}} - 1$. Then as $n \to \infty$, the number of labelled Δ-regular graphs on n vertices is asymptotic to

$$e^{-\lambda-\lambda^2} \frac{(2m)!}{m! 2^m (\Delta!)^m},$$

where $\lambda = (\Delta - 1)/2$. As a consequence of the method we determine the asymptotic distribution of the number of short cycles in graphs with a given degree sequence, and give analogous formulae for hypergraphs.

In 1959 Read [6] determined an exact formula for the number of labelled Δ–regular graphs on n vertices. This formula, whose proof is based on Pólya's enumeration theorem [5], is not easily penetrated. In particular, it seems that only for $\Delta \leqslant 3$ can it be used to find its asymptotic value (see [4, p. 175]). Recently Bender and Canfield [1] gave an asymptotic formula for the number of labelled graphs with given degree sequences by enumerating certain classes of involutions. In this note we offer a somewhat different approach, allowing one to obtain a more general asymptotic formula without much effort and without any reference to an exact formula. In particular, our asymptotic formula holds not only for constant Δ but also if Δ increases rather slowly as $n \to \infty$. As a considerable bonus, the model presented here enables one to give asymptotic formulae for various subclasses of labelled regular graphs. We intend to exploit this possibility in the future.

As customary, we use $A \sim B$ to denote the relation $A/B \to 1$ as $n \to \infty$. Furthermore, we write $(a)_b = a(a-1) \cdots (a-b+1)$. Throughout the proof $c_1, c_2, \ldots$ denote positive constants.

THEOREM 1. *Let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ be natural numbers with $\sum_{i=1}^n d_i = 2m$ even. Suppose*

$$\Delta = d_1 \leqslant (2 \log n)^{\frac{1}{2}} - 1$$

and $m \geqslant \max\{\varepsilon \Delta n, (1+\varepsilon)n\}$ for some fixed $\varepsilon > 0$. Then the number $L(\mathbf{d})$ of labelled graphs with degree sequence $\mathbf{d} = (d_i)_1^n$ satisfies

$$L(\mathbf{d}) \sim e^{-\lambda-\lambda^2} (2m)_m \Big/ \Big\{ 2^m \prod_{i=1}^n d_i! \Big\},$$

where

$$\lambda = \frac{1}{2m} \sum_{i=1}^n \binom{d_i}{2}.$$

PROOF. We shall represent our graphs as images of so called "configurations". Let $W = \bigcup_{j=1}^n W_j$ be a fixed set of $2m = \sum_{j=1}^n d_j$ labelled vertices, where $|W_j| = d_j$. A *configuration* F is a partition of W into m pairs of vertices, called *edges* of F. Clearly there are

$$N = N(m) = \binom{2m}{2}\binom{2m-2}{2} \cdots \binom{2}{2} \Big/ m! = (2m)_m 2^{-m} \quad (1)$$

configurations. Furthermore, if we fix l independent (vertex disjoint) edges then there are

0195-6698/80/040311+06 $01.00/0

Definitions

High Girth and High Chromatic Number

Random Regular Graphs

3-Flow Conjecture

Europ. J. Combinatorics (1980) **1**, 311–316

A Probabilistic Proof of an Asymptotic Formula for the Number of Labelled Regular Graphs

BÉLA BOLLOBÁS

Let Δ and n be natural numbers such that $\Delta n = 2m$ is even and $\Delta \leqslant (2\log n)^{\frac{1}{2}} - 1$. Then as $n \to \infty$, the number of labelled Δ-regular graphs on n vertices is asymptotic to

$$e^{-\lambda-\lambda^2}\frac{(2m)!}{m!2^m(\Delta!)^m},$$

where $\lambda = (\Delta - 1)/2$. As a consequence of the method we determine the asymptotic distribution of the number of short cycles in graphs with a given degree sequence, and give analogous formulae for hypergraphs.

In 1959 Read [6] determined an exact formula for the number of labelled Δ–regular graphs on n vertices. This formula, whose proof is based on Pólya's enumeration theorem [5], is not easily penetrated. In particular, it seems that only for $\Delta \leqslant 3$ can it be used to find its asymptotic value (see [4, p. 175]). Recently Bender and Canfield [1] gave an asymptotic formula for the number of labelled graphs with given degree sequences by enumerating certain classes of involutions. In this note we offer a somewhat different approach, allowing one to obtain a more general asymptotic formula without much effort and without any reference to an exact formula. In particular, our asymptotic formula holds not only for constant Δ but also if Δ increases rather slowly as $n \to \infty$. As a considerable bonus, the model presented here enables one to give asymptotic formulae for various subclasses of labelled regular graphs. We intend to exploit this possibility in the future.

As customary, we use $A \sim B$ to denote the relation $A/B \to 1$ as $n \to \infty$. Furthermore, we write $(a)_b = a(a-1)\cdots(a-b+1)$. Throughout the proof $c_1, c_2, \ldots$ denote positive constants.

THEOREM 1. *Let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ be natural numbers with $\sum_{i=1}^n d_i = 2m$ even. Suppose*

$$\Delta = d_1 \leqslant (2\log n)^{\frac{1}{2}} - 1$$

and $m \geqslant \max\{\varepsilon\Delta n, (1+\varepsilon)n\}$ for some fixed $\varepsilon > 0$. Then the number $L(\mathbf{d})$ of labelled graphs with degree sequence $\mathbf{d} = (d_i)_1^n$ satisfies

$$L(\mathbf{d}) \sim e^{-\lambda-\lambda^2}(2m)_m \Big/ \Big\{2^m \prod_{i=1}^n d_i!\Big\},$$

where

$$\lambda = \frac{1}{2m}\sum_{i=1}^n \binom{d_i}{2}.$$

PROOF. We shall represent our graphs as images of so called "configurations". Let $W = \bigcup_{j=1}^n W_j$ be a fixed set of $2m = \sum_{j=1}^n d_j$ labelled vertices, where $|W_j| = d_j$. A *configuration* F is a partition of W into m pairs of vertices, called *edges* of F. Clearly there are

$$N = N(m) = \binom{2m}{2}\binom{2m-2}{2}\cdots\binom{2}{2}\Big/ m! = (2m)_m 2^{-m} \qquad (1)$$

configurations. Furthermore, if we fix l independent (vertex disjoint) edges then there are

0195-6698/80/040311+06 $01.00/0

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

"Erdős has an amazing ability to match problems with people.

Which is why so many mathematicians benefit from his presence."
– Béla Bollobás

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

"Erdős has an amazing ability to match problems with people.

Which is why so many mathematicians benefit from his presence."
– Béla Bollobás

Europ. J. Combinatorics (1980) **1**, 311–316

A Probabilistic Proof of an Asymptotic Formula for the Number of Labelled Regular Graphs

BÉLA BOLLOBÁS

Let Δ and n be natural numbers such that $\Delta n = 2m$ is even and $\Delta \leqslant (2\log n)^{\frac{1}{2}} - 1$. Then as $n \to \infty$, the number of labelled Δ-regular graphs on n vertices is asymptotic to

$$e^{-\lambda-\lambda^2}\frac{(2m)!}{m!2^m(\Delta!)^m},$$

where $\lambda = (\Delta - 1)/2$. As a consequence of the method we determine the asymptotic distribution of the number of short cycles in graphs with a given degree sequence, and give analogous formulae for hypergraphs.

In 1959 Read [6] determined an exact formula for the number of labelled Δ–regular graphs on n vertices. This formula, whose proof is based on Pólya's enumeration theorem [5], is not easily penetrated. In particular, it seems that only for $\Delta \leqslant 3$ can it be used to find its asymptotic value (see [4, p. 175]). Recently Bender and Canfield [1] gave an asymptotic formula for the number of labelled graphs with given degree sequences by enumerating certain classes of involutions. In this note we offer a somewhat different approach, allowing one to obtain a more general asymptotic formula without much effort and without any reference to an exact formula. In particular, our asymptotic formula holds not only for constant Δ but also if Δ increases rather slowly as $n \to \infty$. As a considerable bonus, the model presented here enables one to give asymptotic formulae for various subclasses of labelled regular graphs. We intend to exploit this possibility in the future.

As customary, we use $A \sim B$ to denote the relation $A/B \to 1$ as $n \to \infty$. Furthermore, we write $(a)_b = a(a-1)\cdots(a-b+1)$. Throughout the proof $c_1, c_2, \ldots$ denote positive constants.

THEOREM 1. *Let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ be natural numbers with $\sum_{i=1}^n d_i = 2m$ even. Suppose*

$$\Delta = d_1 \leqslant (2\log n)^{\frac{1}{2}} - 1$$

and $m \geqslant \max\{\varepsilon\Delta n, (1+\varepsilon)n\}$ for some fixed $\varepsilon > 0$. Then the number $L(\mathbf{d})$ of labelled graphs with degree sequence $\mathbf{d} = (d_i)_1^n$ satisfies

$$L(\mathbf{d}) \sim e^{-\lambda-\lambda^2}(2m)_m \Big/ \Big\{2^m \prod_{i=1}^n d_i!\Big\},$$

where

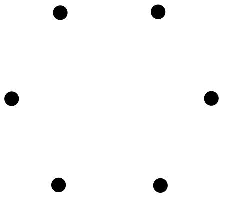
$$\lambda = \frac{1}{2m}\sum_{i=1}^n \binom{d_i}{2}.$$

PROOF. We shall represent our graphs as images of so called "configurations". Let $W = \bigcup_{j=1}^n W_j$ be a fixed set of $2m = \sum_{j=1}^n d_j$ labelled vertices, where $|W_j| = d_j$. A *configuration* F is a partition of W into m pairs of vertices, called *edges* of F. Clearly there are

$$N = N(m) = \binom{2m}{2}\binom{2m-2}{2}\cdots\binom{2}{2}\Big/ m! = (2m)_m 2^{-m} \qquad (1)$$

configurations. Furthermore, if we fix l independent (vertex disjoint) edges then there are

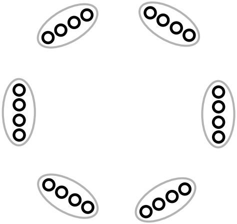
0195-6698/80/040311+06 $01.00/0



$n = 6$

Pairing Model (Bollobás 1980)

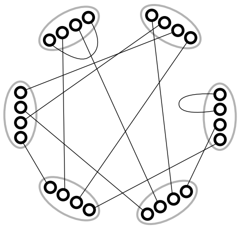
1. *Begin with n vertices.*
2.
3.
4.
5.



$n = 6, d = 4$

Pairing Model (Bollobás 1980)

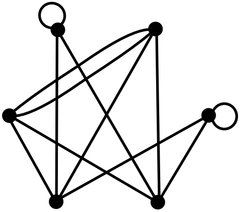
1. *Begin with n vertices.*
2. *Create n "cells," each with d "points." (dn even)*
3.
4.
5.



$n = 6, d = 4$

Pairing Model (Bollobás 1980)

1. *Begin with n vertices.*
2. *Create n "cells," each with d "points." (dn even)*
3. *Form a random perfect matching.*
4.
5.

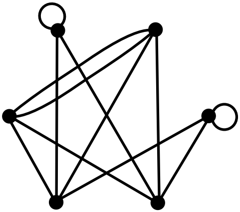


$n = 6, d = 4$

Pairing Model (Bollobás 1980)

1. *Begin with n vertices.*
2. *Create n "cells," each with d "points." (dn even)*
3. *Form a random perfect matching.*
4. *Collapse the cells.*
5.

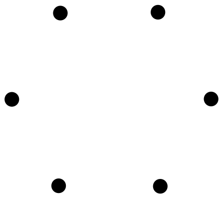
Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture



$n = 6, d = 4$

Pairing Model (Bollobás 1980)

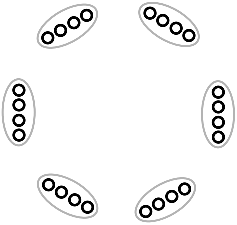
1. *Begin with n vertices.*
2. *Create n "cells," each with d "points." (dn even)*
3. *Form a random perfect matching.*
4. *Collapse the cells.*
5. *If this (multi)graph is not simple, then* ***restart****.*



$n = 6$

Pairing Model (Bollobás 1980)

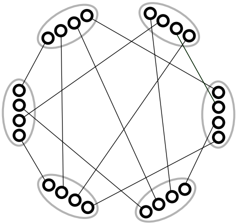
1. *Begin with n vertices.*
2. *Create n "cells," each with d "points." (dn even)*
3. *Form a random perfect matching.*
4. *Collapse the cells.*
5. *If this (multi)graph is not simple, then* ***restart****.*



$n = 6, d = 4$

Pairing Model (Bollobás 1980)

1. *Begin with n vertices.*
2. *Create n "cells," each with d "points." (dn even)*
3. *Form a random perfect matching.*
4. *Collapse the cells.*
5. *If this (multi)graph is not simple, then* ***restart****.*

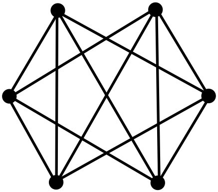


$n = 6, d = 4$

Pairing Model (Bollobás 1980)

1. *Begin with n vertices.*
2. *Create n "cells," each with d "points." (dn even)*
3. *Form a random perfect matching.*
4. *Collapse the cells.*
5. *If this (multi)graph is not simple, then* ***restart****.*

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture



$n = 6, d = 4$

Pairing Model (Bollobás 1980)

1. *Begin with n vertices.*
2. *Create n "cells," each with d "points." (dn even)*
3. *Form a random perfect matching.*
4. *Collapse the cells.*
5. *If this (multi)graph is not simple, then* ***restart****.*

Bootstrap percolation on the random regular graph

József Balogh* and Boris G. Pittel †

December 8, 2005

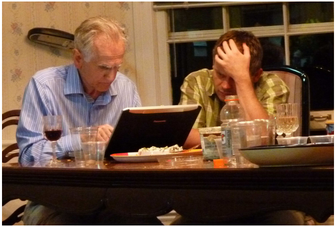
Dedicated to Alan Frieze on the occasion of his 60-th birthday.

Abstract

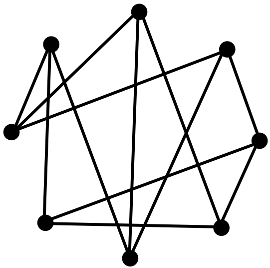
The k-parameter bootstrap percolation on a graph is a model of an interacting particle system, which can also be viewed as a variant of a growth process of a cellular automata with threshold $k \geq 2$. At the start each of the graph vertices is *active* with probability p and *inactive* with probability $1 - p$, independently of other vertices. Presence of active vertices triggers a percolation process controlled by the recursive rule: an active vertex remains active forever, and a currently inactive vertex becomes active when at least k of its neighbors are active. The basic problem is to identify, for a given graph, p^-, p^+ such that for $p < p^-$ ($p > p^+$ resp.) the probability that all vertices are eventually active is very close to 0 (1 resp.). The percolation process is a Markov chain on the space of subsets of the vertex set, which is easy to describe but hard to analyze rigorously in general. We study the percolation on the random d-regular graph, $d \geq 3$, via analysis of the process on its multigraph counterpart. Here, thanks to a "principle of deferred decisions", the percolation dynamics is described by a surprisingly simple Markov chain. Its generic state is formed by the counts of

*University of Illinois at Urbana-Champaign, work was partially done while at the Ohio State University; email: jobal@math.uiuc.edu, research supported in part by NSF grant DMS-0302804 and OTKA grant 049398.

†The Ohio State University; email: bgp@math.ohio-state.edu, research supported in part by NSF grant DMS-0406024.



Bootstrap Percolation on a Random Regular Graph



$d = 3$

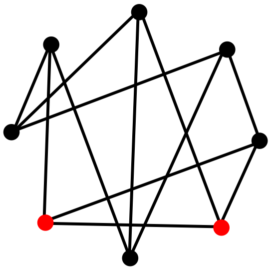
Definitions

High Girth and High Chromatic Number

Random Regular Graphs

3-Flow Conjecture

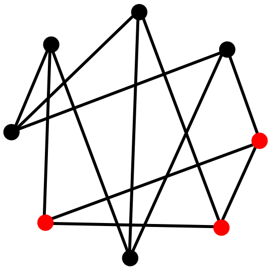
Bootstrap Percolation on a Random Regular Graph



$d = 3, q = \frac{1}{4}$

- Infect vertices independently with some probability q
-
-
-

Bootstrap Percolation on a Random Regular Graph

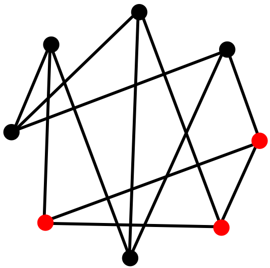


$d = 3, q = \frac{1}{4}$

- Infect vertices independently with some probability q
- Infection spreads to a vertex if $>$ half neighbors are infected
-
-

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

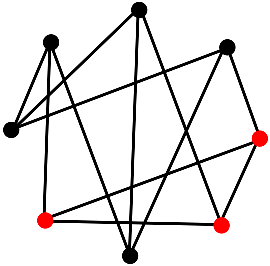
Bootstrap Percolation on a Random Regular Graph



$d = 3, q = \frac{1}{4}$

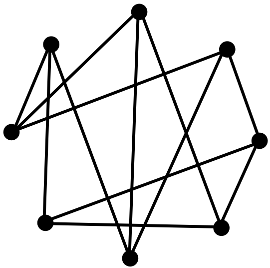
- Infect vertices independently with some probability q
- Infection spreads to a vertex if $>$ half neighbors are infected
- Iterate until stabilizes
-

Bootstrap Percolation on a Random Regular Graph



$d = 3, q = \frac{1}{4}$

- Infect vertices independently with some probability q
- Infection spreads to a vertex if $>$ half neighbors are infected
- Iterate until stabilizes
- Is the whole graph infected?

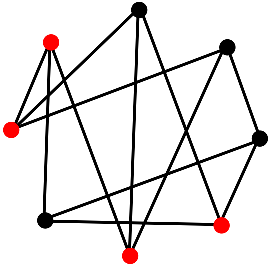


$d = 3$

Bootstrap Percolation on a Random Regular Graph

- Infect vertices independently with some probability q
- Infection spreads to a vertex if $>$ half neighbors are infected
- Iterate until stabilizes
- Is the whole graph infected?

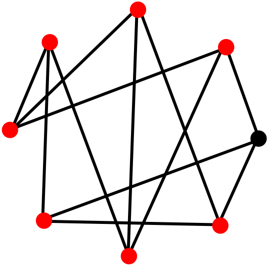
Bootstrap Percolation on a Random Regular Graph



$d = 3, q = \frac{1}{2}$

- Infect vertices independently with some probability q
- Infection spreads to a vertex if $>$ half neighbors are infected
- Iterate until stabilizes
- Is the whole graph infected?

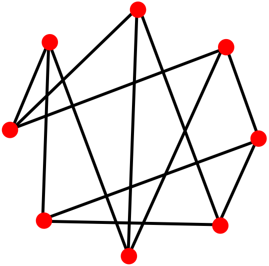
Bootstrap Percolation on a Random Regular Graph



$d = 3, q = \frac{1}{2}$

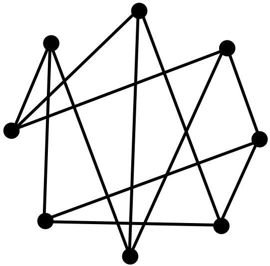
- Infect vertices independently with some probability q
- Infection spreads to a vertex if $>$ half neighbors are infected
- Iterate until stabilizes
- Is the whole graph infected?

Bootstrap Percolation on a Random Regular Graph



$d = 3, q = \frac{1}{2}$

- Infect vertices independently with some probability q
- Infection spreads to a vertex if $>$ half neighbors are infected
- Iterate until stabilizes
- Is the whole graph infected?

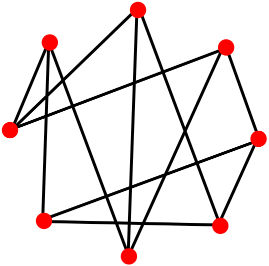


$d = 3, q = 0$

Bootstrap Percolation on a Random Regular Graph

- Is the whole graph infected?
- For $q = 0$, no.

Bootstrap Percolation on a Random Regular Graph



$d = 3, q = 1$

- Is the whole graph infected?
- For $q = 1$, yes.

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Theorem (Balogh and Pittel 2007)

Let $d \geq 3$. For random d-regular graphs, the dissemination threshold is a constant

$$p_d = \frac{d-2}{d-1}$$

asymptotically almost surely (a.a.s.).

$p_3 = \frac{1}{2}, p_4 = \frac{2}{3}$, etc.

where an event $X = X(n)$ holds a.a.s. if $\mathbb{P}[X(n)] \to 1$ as $n \to \infty$

Theorem (Balogh and Pittel 2007)

Let $d \geq 3$. For random d-regular graphs, the dissemination threshold is a constant

$$p_d = \frac{d-2}{d-1}$$

asymptotically almost surely (a.a.s.).

$p_3 = \frac{1}{2}, p_4 = \frac{2}{3}$, etc.

where an event $X = X(n)$ holds a.a.s. if $\mathbb{P}[X(n)] \to 1$ as $n \to \infty$

Definitions

High Girth and High Chromatic Number

Random Regular Graphs

3-Flow Conjecture

3-Flow Conjecture

One of the most famous graph theory conjectures is the Tutte nowhere-zero 3-flow conjecture.

Conjecture (Equivalent Form, Tutte 1966)

Every 4-edge-connected, 5-regular graph has an edge orientation in which every out-degree is either 4 or 1.

Definition (k-edge-connected)

G is k-edge-connected if G remains connected whenever any set of fewer than k edges are removed.

One of the most famous graph theory conjectures is the Tutte nowhere-zero 3-flow conjecture.

Conjecture (Equivalent Form, Tutte 1966)

Every 4-edge-connected, 5-regular graph has an edge orientation in which every out-degree is either 4 or 1.

Definition (k-edge-connected)

G is k-edge-connected if G remains connected whenever any set of fewer than k edges are removed.

One of the most famous graph theory conjectures is the Tutte nowhere-zero 3-flow conjecture.

Conjecture (Equivalent Form, Tutte 1966)

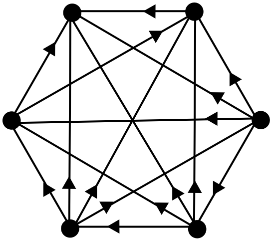
Every 4-edge-connected, 5-regular graph has an edge orientation in which every out-degree is either 4 or 1.

Definition (k-edge-connected)

G is k-edge-connected if G remains connected whenever any set of fewer than k edges are removed.

Conjecture (Equivalent Form, Tutte 1966)

Every 4-edge-connected, 5-regular graph has an edge orientation in which every out-degree is either 4 or 1.



Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Question (Barát and Thomassen 2006)

Does every 4-edge-connected, 4-regular graph have an edge orientation in which every out-degree is either 4 or 1.

Answer: No!

Definitions

High Girth and High Chromatic Number

Random Regular Graphs

3-Flow Conjecture

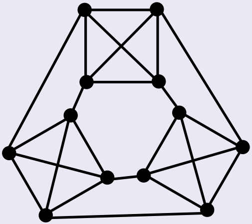
Question (Barát and Thomassen 2006)

Does every 4-edge-connected, 4-regular graph have an edge orientation in which every out-degree is either 4 or 1.

Answer: No!

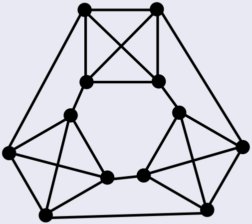
Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Example (Barát and Thomassen 2006)



Pigeonhole!

Example (Barát and Thomassen 2006)



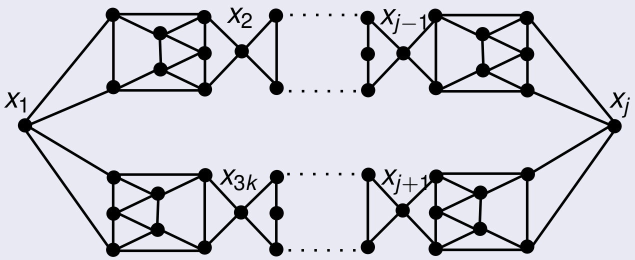
Pigeonhole!

Conjecture (Barát and Thomassen 2006)

If G is a planar 4-edge-connected, 4-regular graph such that $3|e(G)$, then G has an edge orientation in which every out-degree is either 4 or 1.

Definitions
High Girth and High Chromatic Number
Random Regular Graphs
3-Flow Conjecture

Counterexample (Lai 2007)



Theorem (Bollobás 1981, Wormald 1981)

A random d-regular graph is d-edge-connected asymptotically almost surely (a.a.s.).

Theorem (D. and Postle 2016+)

If $3|n$, then a random 4-regular graph on n vertices has an edge orientation in which every out-degree is either 4 or 1 asymptotically almost surely (a.a.s.).

Theorem (Bollobás 1981, Wormald 1981)

A random d-regular graph is d-edge-connected asymptotically almost surely (a.a.s.).

Theorem (D. and Postle 2016+)

If $3|n$, then a random 4-regular graph on n vertices has an edge orientation in which every out-degree is either 4 or 1 asymptotically almost surely (a.a.s.).

Definitions

High Girth and High Chromatic Number

Random Regular Graphs

3-Flow Conjecture

Theorem (D. and Postle 2016+)

If $3|n$, then a random 4-regular graph on n vertices has an edge orientation in which every out-degree is either 4 or 1 asymptotically almost surely (a.a.s.).

Idea:

- the proof is very technical, the standard technique does not apply
- instead use the small subgraph conditioning method of Robinson and Wormald

Theorem (D. and Postle 2016+)

If $3|n$, then a random 4-regular graph on n vertices has an edge orientation in which every out-degree is either 4 or 1 asymptotically almost surely (a.a.s.).

Idea:

- the proof is very technical, the standard technique does not apply
- instead use the small subgraph conditioning method of Robinson and Wormald

Theorem (D. and Postle 2016+)

If $3|n$, then a random 4-regular graph on n vertices has an edge orientation in which every out-degree is either 4 or 1 asymptotically almost surely (a.a.s.).

Idea:

- the proof is very technical, the standard technique does not apply
- instead use the small subgraph conditioning method of Robinson and Wormald

Definitions

High Girth and High Chromatic Number

Random Regular Graphs

3-Flow Conjecture

Thank you for listening!