# On Star Decompositions of Random Regular Graphs

**Michelle Delcourt** and Luke Postle

ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

UNIVERSITY OF WATERLOO

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# Background

1. Jaeger's Conjecture
2. Random Versions
3. Barát and Thomassen's Conjecture

# Jaeger's Conjecture

**Conjecture (Tutte 1966)**

*Every 4-edge-connected graph has a nowhere-zero 3-flow.*

Equivalently

**Conjecture**

*Every 4-edge-connected, 5-regular graph has an edge orientation in which every out-degree is either 4 or 1.*

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More generally

**Conjecture (Jaeger 1988)**

*Every $4k$-edge-connected, $(4k+1)$-regular graph has a mod $(2k+1)$-orientation, that is, an edge orientation in which every out-degree is either $3k+1$ or $k$.*

**Theorem (L. M. Lovász, Thomassen, Wang, Zhu, 2013)**

*For every odd $k \geq 3$, every $(3k-3)$-edge-connected graph has a mod $k$-orientation.*

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Using the small subgraph conditioning method of Robinson and Wormald,

**Theorem (Prałat and Wormald 2015+)**

*Tutte's 3-flow conjecture holds asymptotically almost surely for random 5-regular graphs.*

Using spectral techniques, (expander mixing lemma)

**Theorem (Alon and Prałat 2011)**

*For large $k$, Jaeger's conjecture holds asymptotically almost surely for random $(4k+1)$-regular graphs.*

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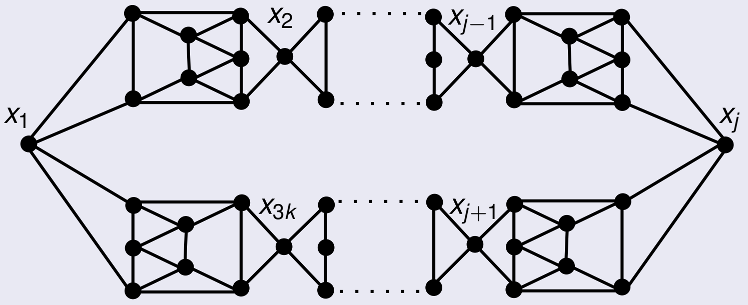
# Barát and Thomassen's Conjecture

**Conjecture (Barát and Thomassen 2006)**

*If G is a planar 4-edge-connected, 4-regular graph such that* $3|e(G)$, *then G has a claw decomposition.*

# Barát and Thomassen's Conjecture

## Counterexample (Lai 2007)



# Barát and Thomassen's Conjecture

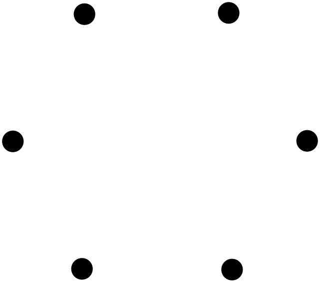
**Theorem (D. and Postle 2015+)**

*If $3|n$, then a random 4-regular graph on n vertices has an $S_3$ decomposition asymptotically almost surely (a.a.s.).*

# Random Regular Graphs

1. Configuration Model $P_{n,d}$
2. Orienting Edges
3. Signatures

# Configuration Model $P_{n,d}$



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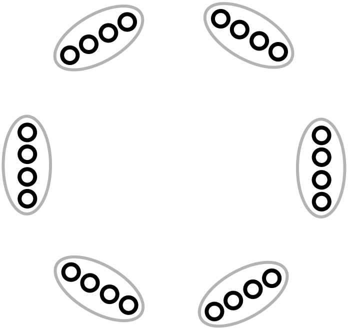

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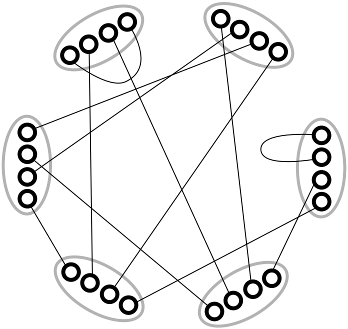

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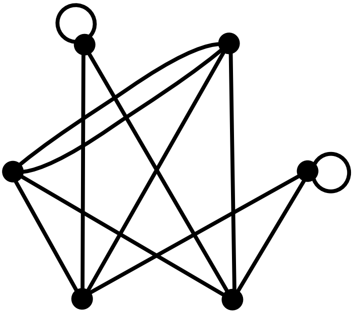

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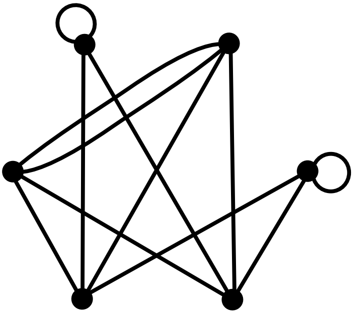

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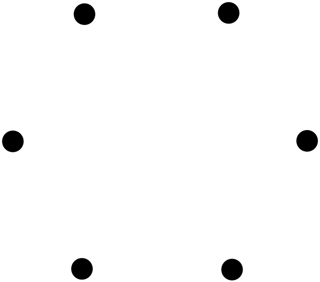

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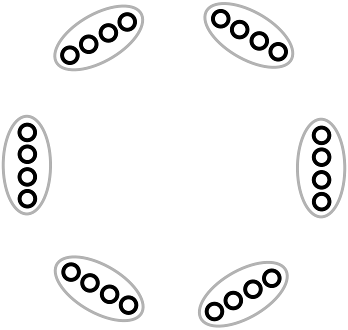

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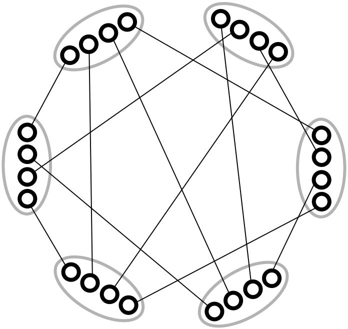

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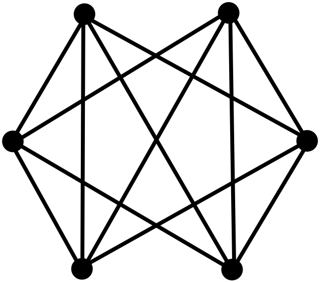

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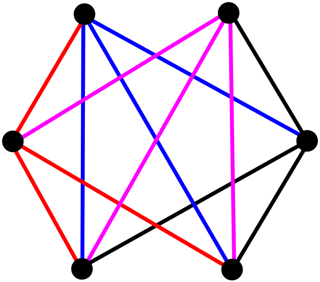
2. Orienting Edges
3. Signatures

# Orienting Edges



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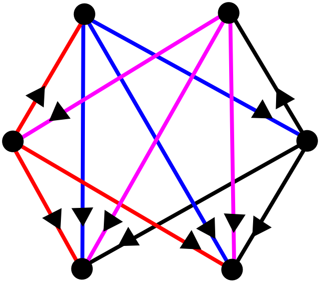
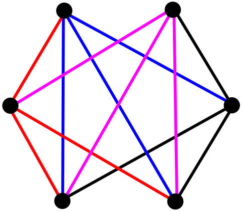
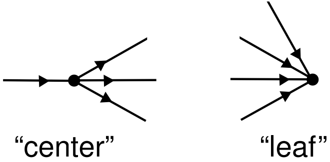


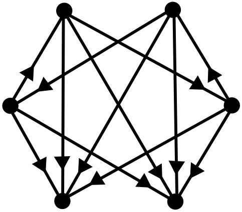
An $S_3$-**decomposition** of a graph $G$ is a partition of $E(G)$ into disjoint copies of $S_3$.

For 4-regular graphs, $S_3$-decompositions are equivalent to orientations with out-degrees equal to 0 or 3.

# Orienting Edges





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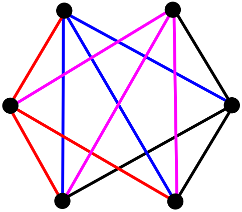

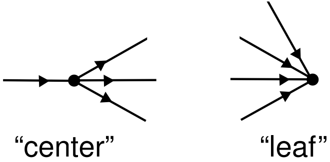


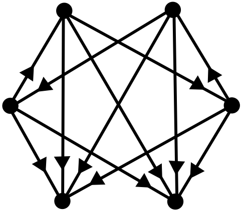

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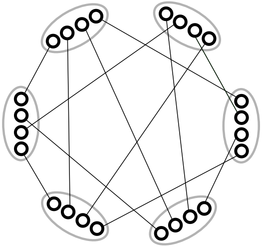

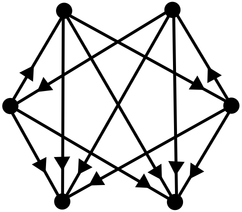

What does this mean in the configuration model?

We assign points

“in” “out”

according to some rules.

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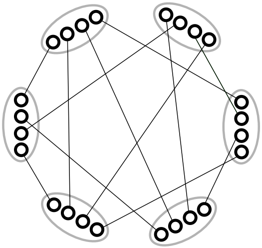

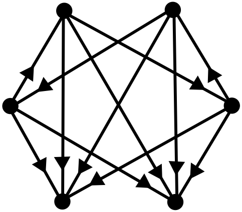

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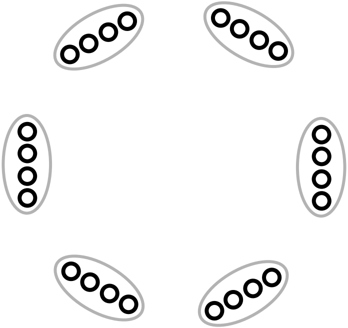

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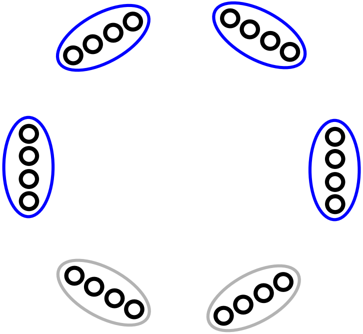

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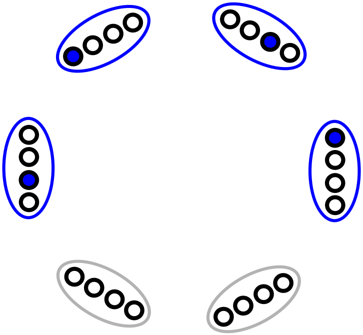

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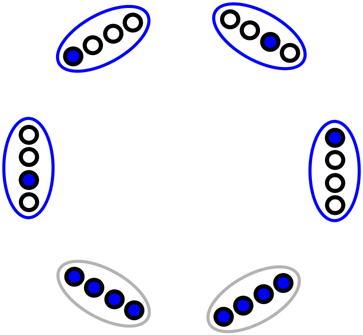

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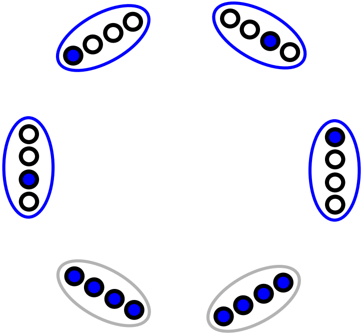

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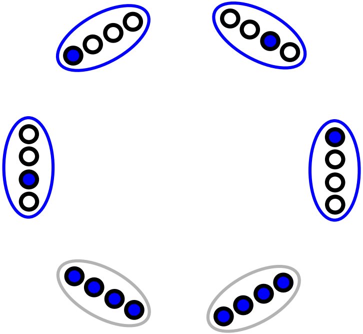

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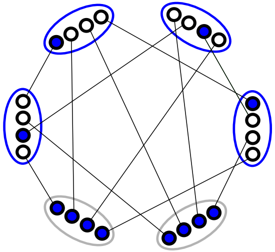

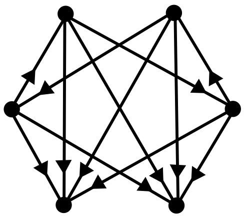

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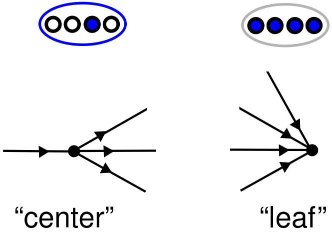

# Finding $S_3$-Decompositions

1. Main Result
2. Small Subgraph Conditioning Method

# Main Result

**Theorem (D. and Postle 2015+)**

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# Main Result

$Y = Y(n) := \# \; S_3$-decompositions of a random element of $P_{n,4}$.

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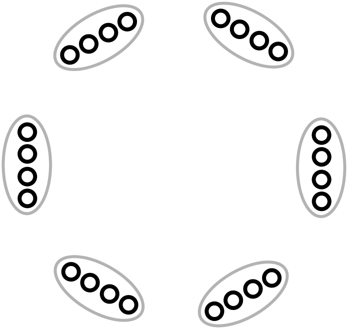

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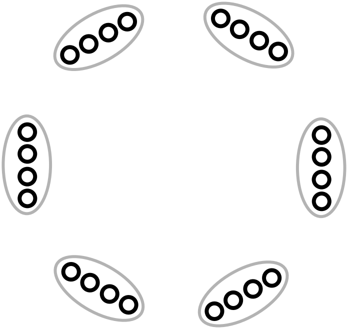

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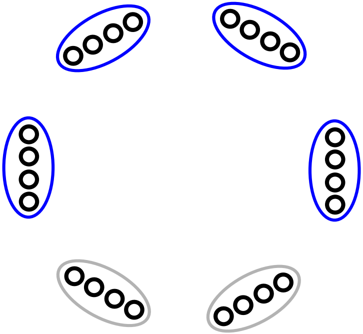

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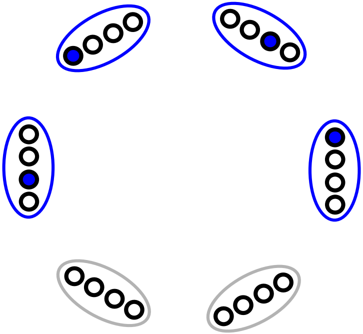

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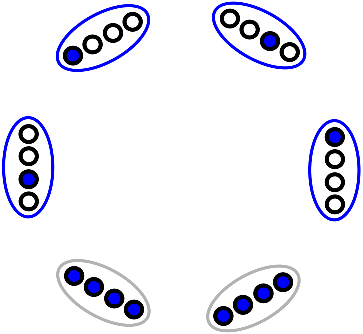

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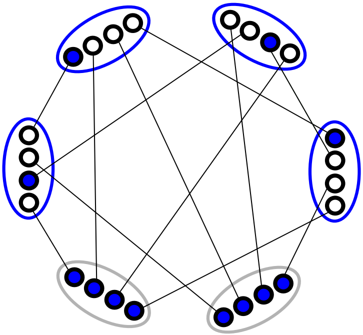

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- $\left(\frac{4n}{2}\right)! = (2n)!$ ways to match "in" points to "out" points.

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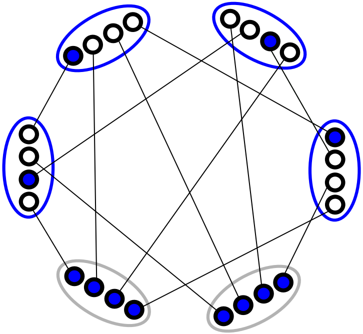

$M(4n)$ is the number of perfect matchings on $4n$ points.

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$Y = Y(n) := \#\ S_3$-decompositions of a random element of $P_{n,4}$.

Using Stirling's approximation,

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We would like to use the second moment method.

**Lemma**

*If $Y$ is a non-negative random variable and $\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]^2} \to 0$ as $n \to \infty$, then a.a.s. $Y > 0$.*

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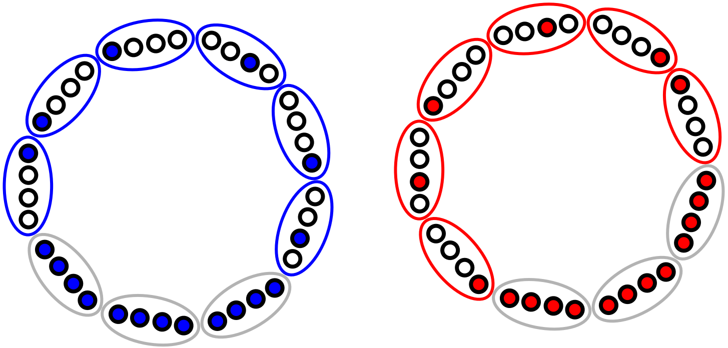

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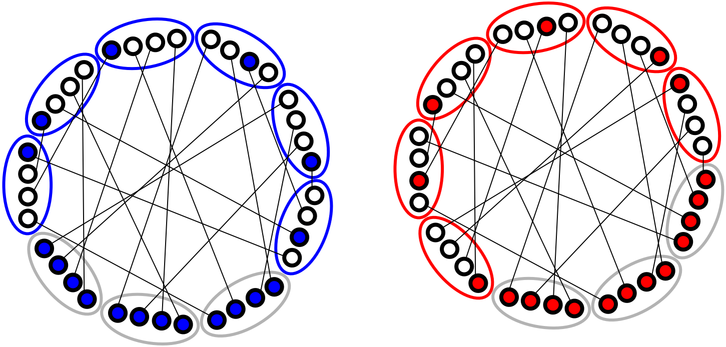
To calculate $\mathbb{E}[Y^2]$, we fix two signatures, say $S_1$ and $S_2$, and see how many configurations jointly they extend to.



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Recall

$$\mathbb{E}[Y] \sim \frac{3}{\sqrt{2}} \left(\frac{27}{16}\right)^{n/3}.$$

Unfortunately

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# Finding $S_3$-Decompositions

1. Main Result
2. Small Subgraph Conditioning Method

# Small Subgraph Conditioning Method

We will try the small subgraph conditioning method of Robinson and Wormald.

When this method works, conditioning on small subgraph counts alters $\mathbb{E}[Y]$ by a constant factor.

"Mysteriously" by conditioning on the numbers of enough small subgraphs, we can reduce $Var[Y]$ to any small fraction of $\mathbb{E}[Y]^2$.

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*For $d$ fixed, let $X_j$ denote the number of cycles of length $j$ in the random multigraph resulting from a pairing in $P_{n,d}$.*
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We need to show that for each $j \geq 1$,

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and more generally (easy generalization)

$$\frac{\mathbb{E}\left[Y[X_1]_{\ell_1} \dots [X_j]_{\ell_j}\right]}{\mathbb{E}[Y]} \to \prod_{i=1}^{j} \left(\lambda_i \left(1 + \delta_i\right)\right)^{\ell_i}$$

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For each $j \geq 1$,

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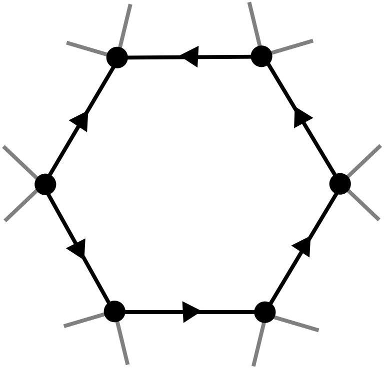

An oriented cycle with $j$ vertices has:

$s$ sinks,
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How many oriented cycles of length $j$ with $s$ sources are there?

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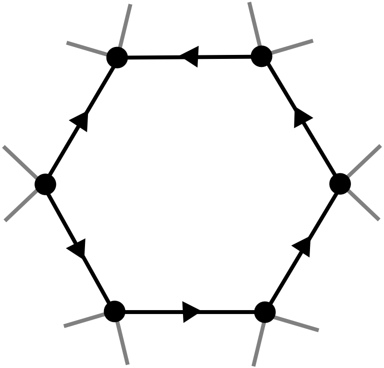

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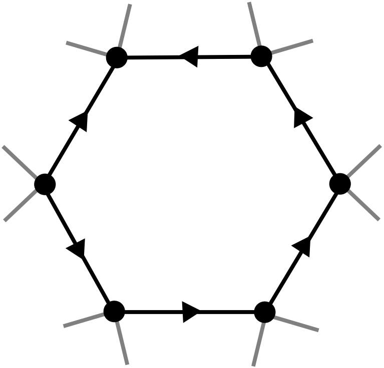

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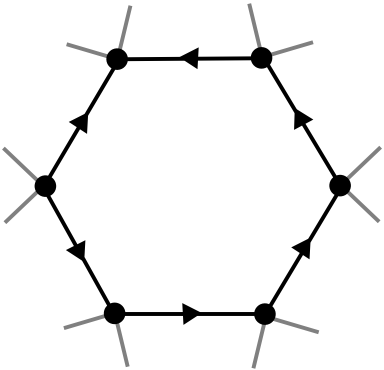

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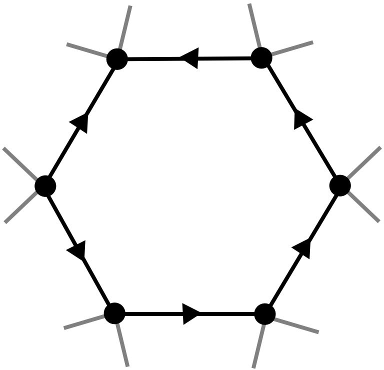

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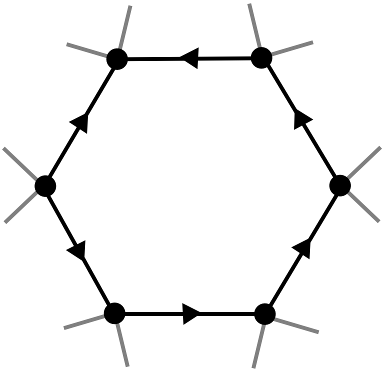

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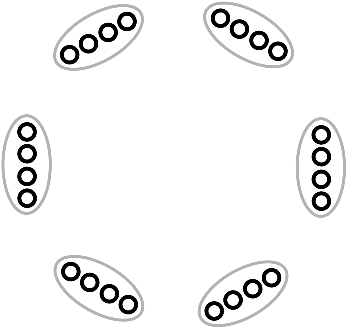

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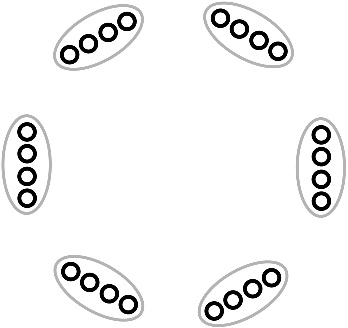

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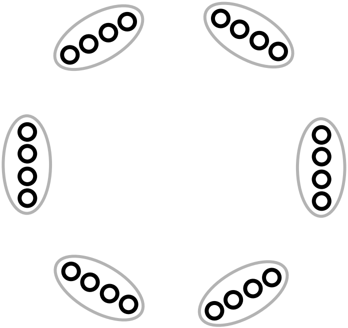

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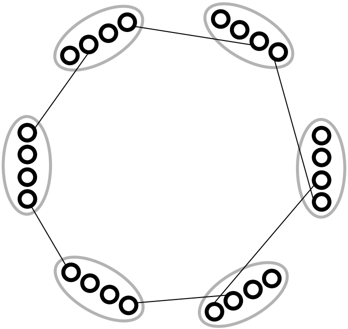

$$\frac{[n]_j}{j}\binom{j}{2s}(4\cdot 3)^j.$$

- $\binom{n}{j}$ choices of $j$ cells
- $\frac{(j-1)!}{2}$ choices of cycle
- $(4\cdot 3)^j$ matchings
- $\binom{j}{j-2s} = \binom{j}{2s}$ choices of non-sink non-source cells
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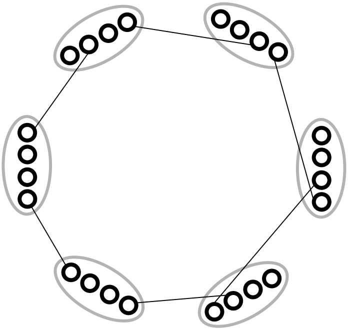

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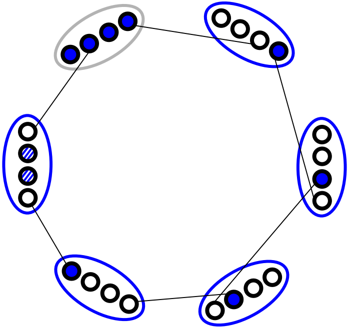

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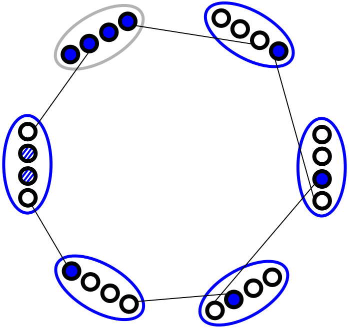

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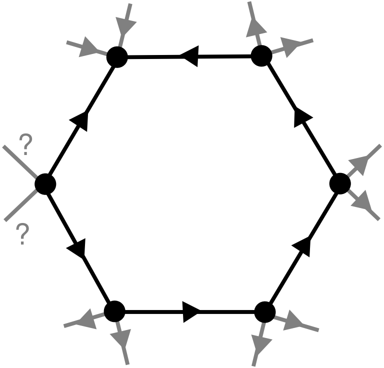


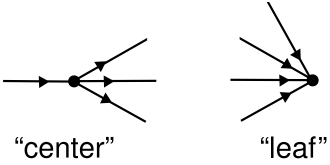


sinks are leaves,
sources are centers, and
non-sinks, non-sources
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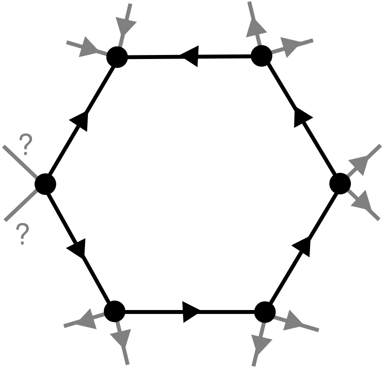


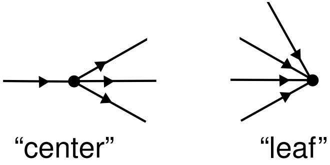


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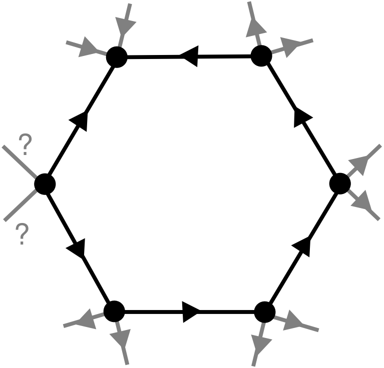


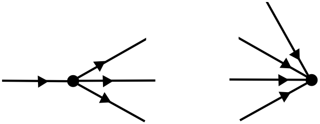

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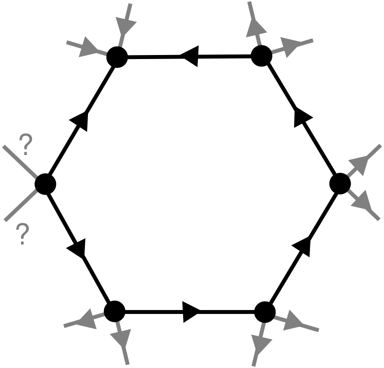


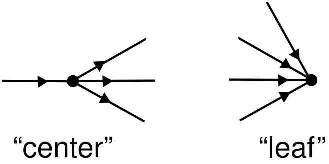


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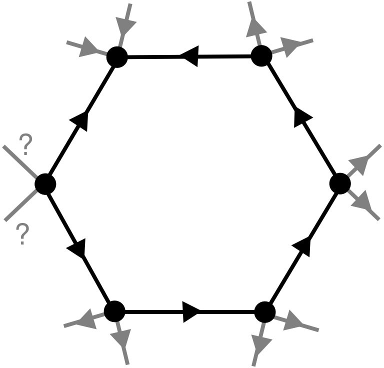
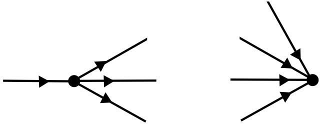


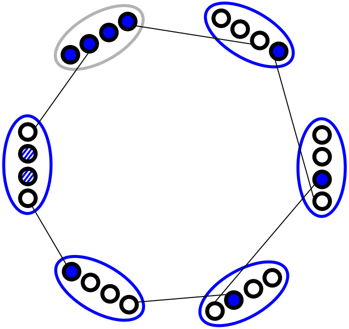



"center" "leaf"

There are $s$ leaves and $j - s$ centers.

# Small Subgraph Conditioning Method

$$\mathbb{E}[YX_j] = \frac{1}{M(4n)} \sum_{s=0}^{\lfloor j/2 \rfloor} \frac{[n]_j}{j} \binom{j}{2s} (4 \cdot 3)^j \cdot 2^s \binom{n-j}{\frac{2n}{3} - j + s} 4^{\frac{2n}{3} - j + s} (2n-j)!$$



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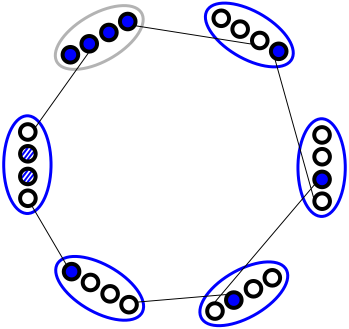

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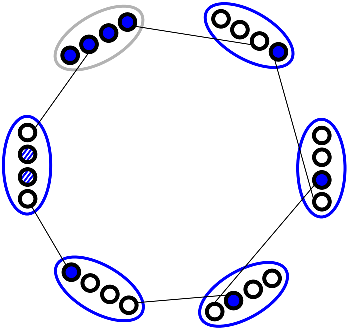

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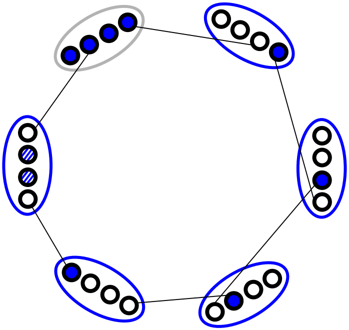

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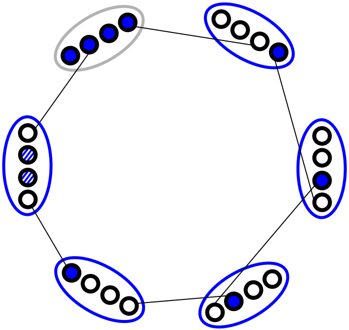

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Note that $\binom{j}{2s}$ is the coefficient of $x^{2s}$ in $q(x) := (1+x)^j$.

$$\frac{\mathbb{E}[YX_j]}{\mathbb{E}[Y]} \sim \frac{1}{j}\sum_{s=0}^{\lfloor j/2 \rfloor}\binom{j}{2s}2^{2s} = \frac{1}{j}\cdot\frac{(q(2)+q(-2))}{2}$$

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$$= \lambda_j\left(1+\left(-\frac{1}{3}\right)^j\right).$$

Therefore, let

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# Small Subgraph Conditioning Method

## Theorem (Robinson and Wormald 1992)

*Let $\lambda_j > 0$ and $\delta_j \geq -1$ be real, $j \geq 1$. Suppose for each $n$ there are non-negative random variables $X_j = X_j(n)$, $j \geq 1$, and $Y = Y(n)$ defined on the same probability space such that $X_j$ is integer valued and $\mathbb{E}[Y] > 0$ (for $n$ sufficiently large). Furthermore, suppose*

1. *For each $j \geq 1$, $X_1, X_2, \ldots, X_j$ are asymptotically independent Poisson random variables with*
$$\mathbb{E}[X_i] \to \lambda_i, \text{ for all } i \in [j];$$
2. $$\frac{\mathbb{E}\left[Y[X_1]_{\ell_1} \ldots [X_j]_{\ell_j}\right]}{\mathbb{E}[Y]} \to \prod_{i=1}^{j} (\lambda_i (1 + \delta_i))^{\ell_i}$$
*for any fixed $\ell_1, \ldots, \ell_j$ where $[X]_\ell$ is the falling factorial;*
3. $\sum_i \lambda_i \delta_i^2 < \infty$;
4. $$\frac{\mathbb{E}[Y(n)^2]}{\mathbb{E}[Y(n)]^2} \leq \exp\left(\sum_i \lambda_i \delta_i^2\right) + o(1) \text{ as } n \to \infty.$$

*Then,*
$$\mathbb{P}[Y(n) > 0] = \exp\left(-\sum_{\delta_i = -1} \lambda_i\right) + o(1).$$

# Small Subgraph Conditioning Method

Then

$$\exp\left(\sum_{j\geq 1}\lambda_j\delta_j^2\right) = \exp\left(\sum_{j\geq 1}\frac{3^j}{2\cdot j}\left(-\frac{1}{3}\right)^{2j}\right) = \exp\left(\frac{1}{2}\sum_{j\geq 1}\frac{1}{j\cdot 3^j}\right)$$

Using $\sum_{i\geq 1}\frac{x^i}{i} = -\ln(1-x)$ for all $-1 < x < 1$,

$$\exp\left(\sum_{j\geq 1}\lambda_j\delta_j^2\right) = \exp\left(\frac{1}{2}(-\ln(2/3))\right) = \sqrt{\frac{3}{2}}.$$

By the small subgraph conditioning method a.a.s. $Y > 0$, because

$$\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]^2} \sim \sqrt{\frac{3}{2}}.$$

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# Small Subgraph Conditioning Method

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$$\exp\left(\sum_{j\geq 1}\lambda_j\delta_j^2\right) = \exp\left(\sum_{j\geq 1}\frac{3^j}{2\cdot j}\left(-\frac{1}{3}\right)^{2j}\right) = \exp\left(\frac{1}{2}\sum_{j\geq 1}\frac{1}{j\cdot 3^j}\right)$$

Using $\sum_{i\geq 1}\frac{x^i}{i} = -\ln(1-x)$ for all $-1 < x < 1$,

$$\exp\left(\sum_{j\geq 1}\lambda_j\delta_j^2\right) = \exp\left(\frac{1}{2}(-\ln(2/3))\right) = \sqrt{\frac{3}{2}}.$$

By the small subgraph conditioning method a.a.s. $Y > 0$, because

$$\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]^2} \sim \sqrt{\frac{3}{2}}.$$

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# Thank you for listening!