Discrete Bernoulli Convolutions

Taking the Convoluted out of Bernoulli Convolutions

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This is joint work with Neil J. Calkin, Julia Davis, Zebediah Engberg, Jobby Jacob, and Kevin James.

Motivation

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A Bernoulli convolution for $0 < q < 1$ is the convolution

$$\mu_q(X) = b(X) * b(X/q) * b(X/q^2) * \dots$$

where b is the discrete Bernoulli measure concentrated at 1 and -1 each with weight $\frac{1}{2}$.

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$$F(t) = \frac{1}{2}F\left(\frac{t-1}{q}\right) + \frac{1}{2}F\left(\frac{t+1}{q}\right)$$

for t on the interval $I_q := \left[\frac{-1}{1-q}, \frac{1}{1-q}\right]$.

There is a unique bounded solution $F_q(t)$, the distribution function of μ_q, $F_q(t) = \mu_q([-\infty, t])$.

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Jessen and Wintner showed that $F_q(t)$ is either absolutely continuous or purely singular. The major question is:

Which values of q make $F_q(t)$ absolutely continuous?

When $0 < q < \frac{1}{2}$, Kershner and Wintner have shown that $F_q(t)$ is always singular. For these values of q, the solution $F_q(t)$ is an example of a so called *Cantor function*, a function that is constant almost everywhere.

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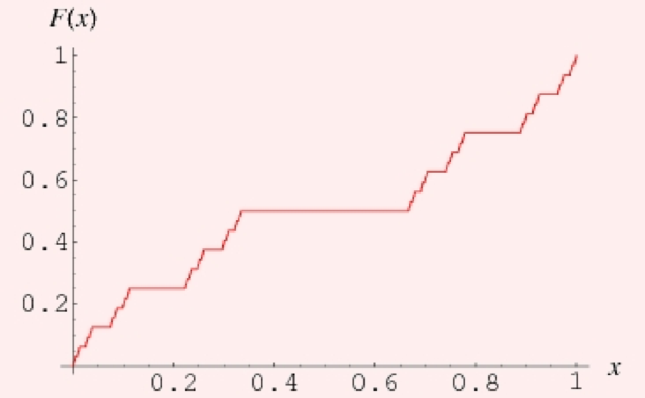

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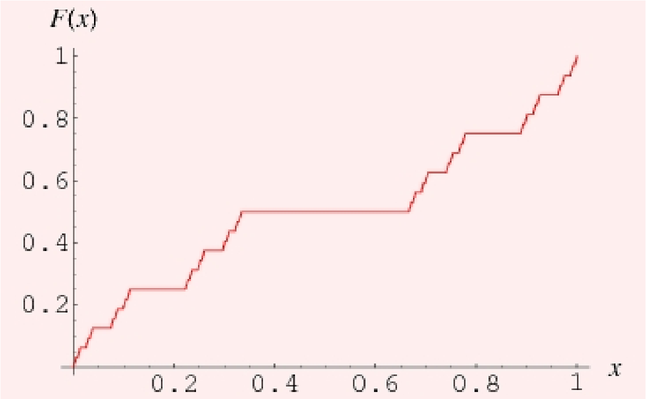

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The case when $q > \frac{1}{2}$ is much harder and more interesting.

In 1939 Erdős showed that if q is of the form $q = \frac{1}{\theta}$ with θ a *Pisot number*, then $F_q(t)$ is again singular.

A *Pisot number* is an algebraic integer greater than 1 in absolute value, whose conjugates are all less than 1 in absolute value.

For example, the golden ratio $\tau = \frac{(1+\sqrt{5})}{2}$ is a Pisot number.

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If $F_q(t)$ is absolutely continuous, then one may consider its derivative $f_q(t) := F_q'(t)$,

$$f(t) = \frac{1}{2q} f\left(\frac{t-1}{q}\right) + \frac{1}{2q} f\left(\frac{t+1}{q}\right).$$

The existence of an absolutely continuous solution $F_q(t)$ is equivalent to the the existence of an $L^1(I_q)$ solution $f_q(t)$ to $f(t)$.

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Motivated by work of Girgensohn in 2007, for $q = \frac{2}{3}$, Calkin shifted the interval $I_q = [-3, 3]$ to $[0, 1]$ for simplicity and considered transform $T : L^1([0, 1]) \longrightarrow L^1([0, 1])$ where

$$T : f(x) \longmapsto \frac{3}{4} f\left(\frac{3x}{2}\right) + \frac{3}{4} f\left(\frac{3x-1}{2}\right).$$

Started with an arbitrary initial function $f^0(t) \in L^1(I_q)$ and iterate the transform T to gain a sequence of functions $f^0, f^1, f^2, \ldots$

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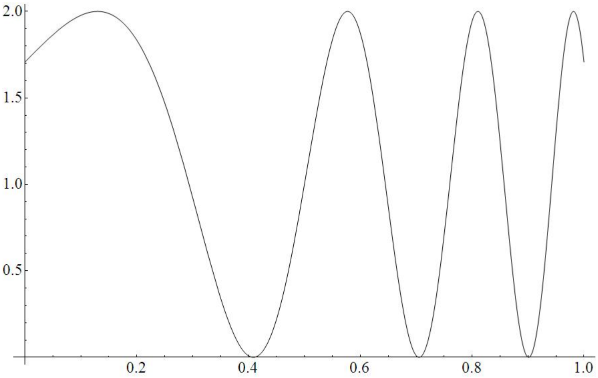

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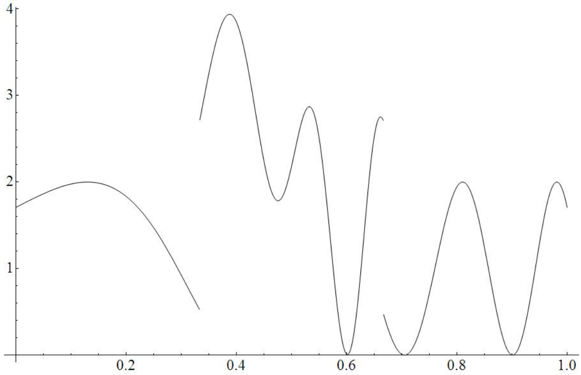

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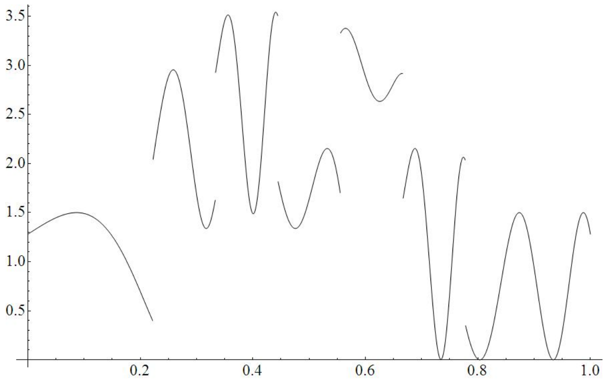

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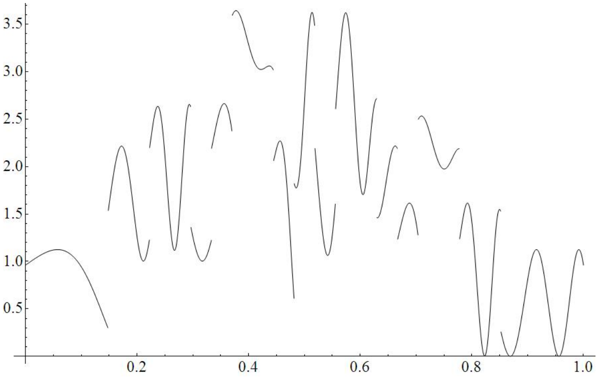

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Intuitively, this transform takes two scaled copies of $f(x)$: one on the interval $\left[0, \frac{2}{3}\right]$ and the other on $\left[\frac{1}{3}, 1\right]$, and adds them.

The scaling factor of $\frac{3}{4}$ gives us that

$$\int_0^1 f(x)dx = \int_0^1 Tf(x)dx.$$

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Duplicate, Shift, Add

Instead of viewing T as a transform on $[0, 1]$, we consider a combinatorial analogue.

Consider the two maps $\text{dup}_n, \text{shf}_n : \mathbb{R}^n \longrightarrow \mathbb{R}^{3n}$ defined by

$$\text{dup}_n : (a_1, a_2, ..., a_{n-1}, a_n) \longmapsto (a_1, a_1, a_2, a_2, ..., a_{n-1}, a_{n-1}, a_n, a_n, \overbrace{0, ..., 0}^{n \text{ times}})$$

$$\text{shf}_n : (a_1, a_2, ..., a_{n-1}, a_n) \longmapsto (\overbrace{0, ..., 0}^{n \text{ times}}, a_1, a_1, a_2, a_2, ..., a_{n-1}, a_{n-1}, a_n, a_n).$$

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Duplicate, Shift, Add

The combinatorial analogue of T on $[0, 1]$ with $f^0(x) = 1$ is provided by the sequences

$$B_0 = (1) \text{ and}$$

$$B_{n+1} = \text{dup}_n(B_n) + \text{shf}_n(B_n).$$

$$\begin{array}{ccccc}
1 & \longrightarrow & 1 & 1 & \\
 & & & 1 & 1 \\
\hline
 & & 1 & 2 & 1
\end{array}$$

$$\begin{array}{cccccccccccc}
1 & 2 & 1 & \longrightarrow & 1 & 1 & 2 & 2 & 1 & 1 & & \\
 & & & & & & & 1 & 1 & 2 & 2 & 1 & 1 \\
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1\ 1\ 2\ 3\ 2\ 3\ 2\ 1\ 1 \longrightarrow 1\ 1\ 1\ 1\ 2\ 2\ 3\ 3\ 2\ 2\ 3\ 3\ 2\ 2\ 1\ 1\ 1\ 1 \\
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A Useful Property

The fact that B_n has a total of 3^n terms follows directly from the definition of dup_n and shf_n.

The average value of B_n, $\mu(B_n) = \left(\frac{4}{3}\right)^n$.

The first few maximum values of B_n, m_n, are $1, 2, 3, 4, 6, 8, 11, 14, 18, 25, 33, 43, 56, 75, 99, 131, 176, 232, \ldots$

Does m_n also grow like $\left(\frac{4}{3}\right)^n$?

If $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, then $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

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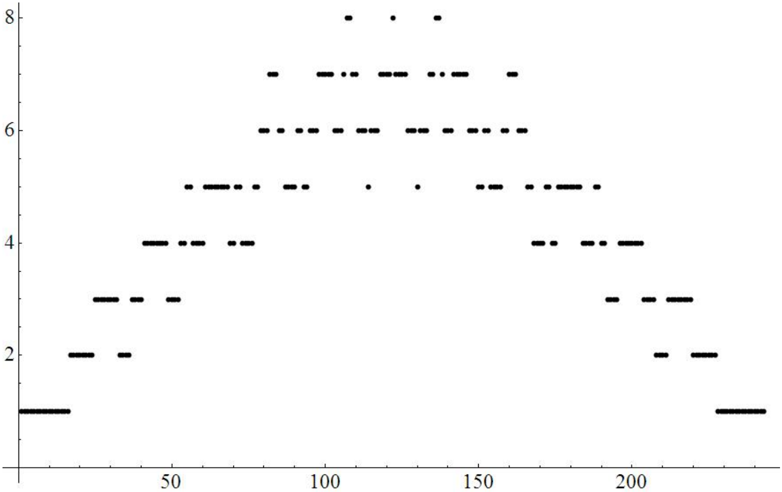
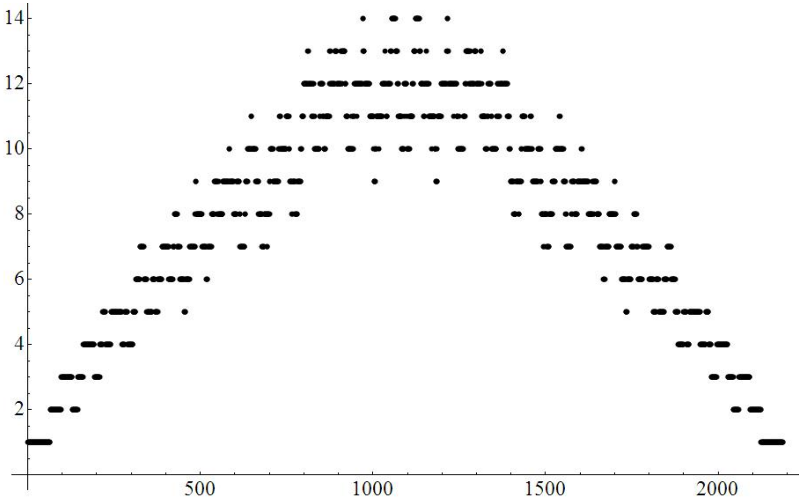
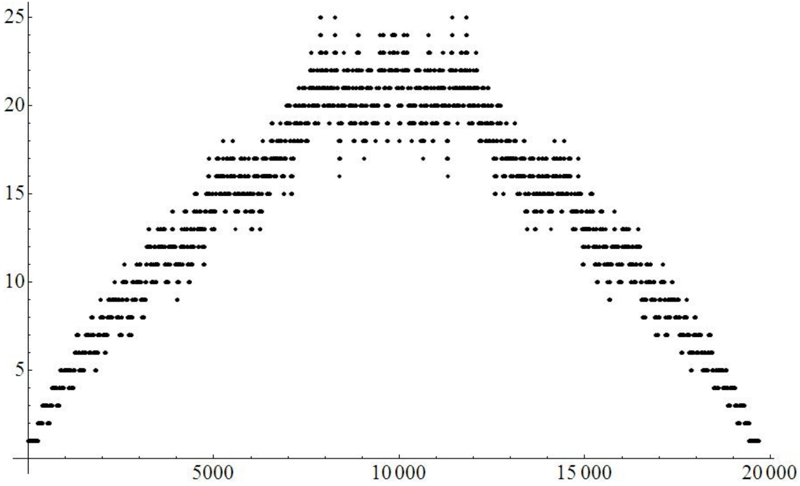


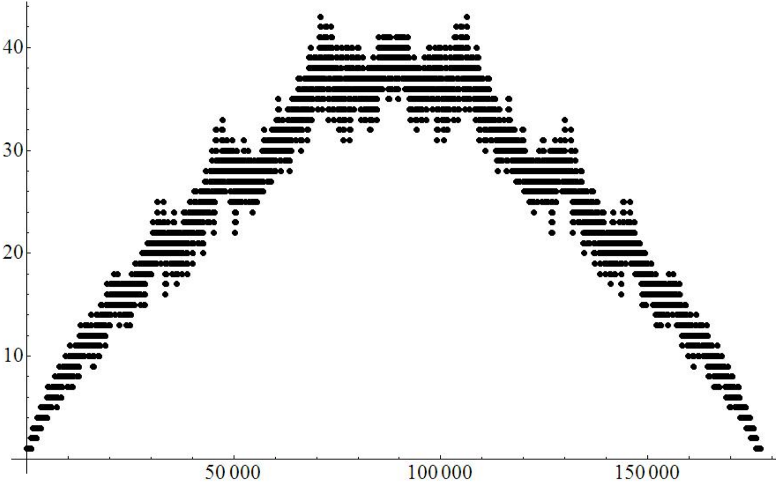
Index versus B_7 entry



Index versus B_9 entry



Index versus B_{11} entry



Polynomial Approach

Translating DSA as a Polynomial Recursion

Consider the polynomial $p_n(x) := b_0 + b_1 x + ... + b_t x^t$ where $B_n = (b_0, b_1, ..., b_t)$ is the Bernoulli sequence on level n where $t = 3^n - 1$.

We see that the duplication $b_0, b_0, b_1, b_1, ..., b_r, b_r$ corresponds to the polynomial $(1 + x)p_n(x^2)$. Shifting the sequence 3^n places to the right corresponds to multiplication by x^{3^n}.

Thus, for $p_0(x) = 1$ we have the recurrence

$$p_{n+1}(x) = (1 + x)p_n(x^2)\left(1 + x^{3^n}\right).$$

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This formula allows us to explicitly solve for $p_n(x)$.

Theorem

The polynomials $p_n(x)$ satisfy

$$p_n(x) = \prod_{i=0}^{n-1}\left(1 + x^{2^i}\right)\prod_{j=0}^{n-1}\left(1 + x^{2^{n-1}(3/2)^j}\right).$$

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$$q_n(x) = \prod_{i=0}^{n-1}\left(1 + x^{2^i}\right) \qquad s_n(x) = \prod_{\substack{1\le j\le n-1 \\ j \text{ odd}}}\left(1 + x^{2^{n-1}(3/2)^j}\right)$$

$$r_n(x) = \prod_{\substack{1\le j\le n-1 \\ j \text{ even}}}\left(1 + x^{2^{n-1}(3/2)^j}\right) = \prod_{j=1}^{\lfloor (n-1)/2\rfloor}\left(1 + x^{2^{n-1}(9/4)^j}\right).$$

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Consider the polynomial $q_n(x) r_n(x)$. Because $9/4 > 2$, we have distinct powers of x when we expand $q_n(x) r_n(x)$.

In other words, the coefficients are all either 0 or 1. Hence the coefficients of $q_n(x) r_n(x)(1 + x^{2^{n-1}})$ are all either 0, 1, or 2.

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In other words, the coefficients are all either 0 or 1. Hence the coefficients of $q_n(x) r_n(x)(1 + x^{2^{n-1}})$ are all either 0, 1, or 2.

Clever Factoring

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In particular, the coefficients are bounded. On the other hand, there are at most $n/2$ terms in the product defining $s_n(x)$.

Hence there are at most $2^{n/2}$ nonzero terms in the polynomial $s_n(x)$ since we have 2 choices from each term in the product.

Therefore the coefficients of $p_n(x)$ are all $O(2^{n/2}) = O((\sqrt{2})^n)$.

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Better Bound

Normalize

Seeing that our sequence on level n has length 3^n, we naturally index it by the first 3^n nonnegative integers.

In certain circumstances, it is advantageous to normalize the indexing in such a way that each index is on the interval $[0, 1]$.

To this end, we can simply take the image of $k \in \{0, 1, 2, ..., 3^n - 1\}$ under the map $k \mapsto k/3^n$.

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Notation

Let $g_n(x)$ denote the n^{th} level Bernoulli sequence where now $x \in [0,1]$. In other words,

$$g_n\left(\frac{k}{3^n}\right) = b_k \qquad \text{for } k = 0, 1, ...3^n - 1.$$

For a subset $S \subset [0,1]$, we define

$$\Gamma_n(S) = \max_{x \in \overline{S}} g_n(x)$$

where $\overline{S} = S \cap \left\{0, \frac{1}{3^n}, \frac{2}{3^n}, \ldots, \frac{3^n - 1}{3^n}\right\}$.

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An Example

We now walk through an example to demonstrate our algorithm.

Each entry on level n can be written as a sum of entries of previous levels. In this particular example we write each entry on level n as a sum of entries on level $n - 3$.

We break up the interval $[0, 1]$ into subintervals of length $1/81$. Let's see what we get.

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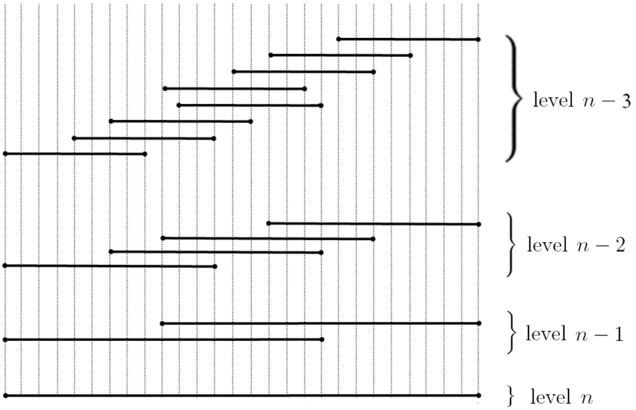
Pullback diagram



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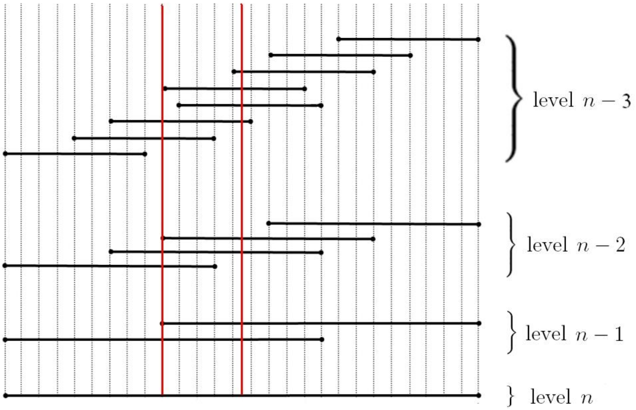

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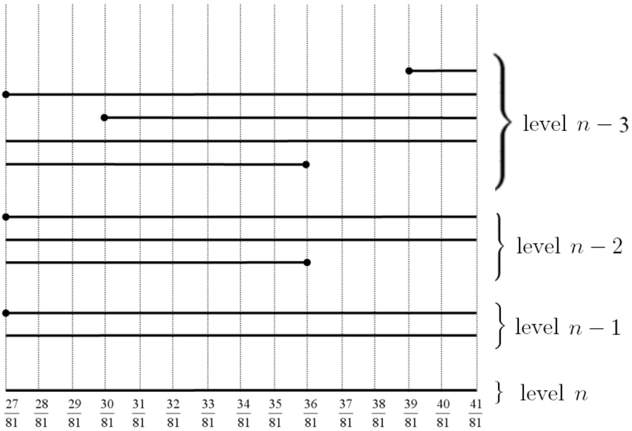

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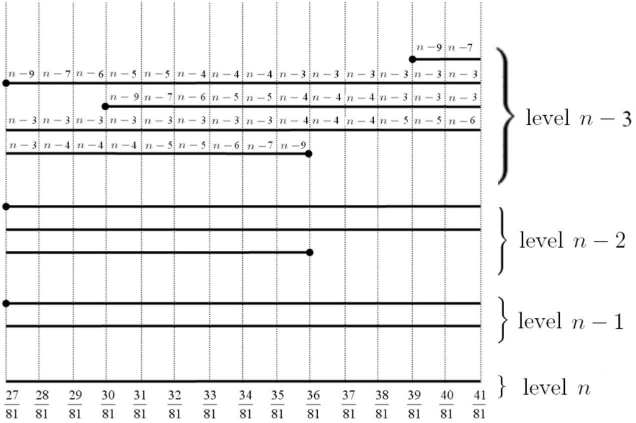

Largest real root

Interval	Polynomial	Largest real root
1	$x^n - 2x^{n-3} - x^{n-9}$	1.301688030...
2	$x^n - x^{n-3} - x^{n-4} - x^{n-7}$	1.288452726...
3	$x^n - x^{n-3} - x^{n-4} - x^{n-6}$	1.304077155...
4	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-9}$	1.349240712...
5	$x^n - x^{n-3} - x^{n-7} - 2x^{n-5}$	1.342242489...
6	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-6}$	1.380277569...
7	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-6}$	1.380277569...
8	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-7}$	1.366811194...
9	$x^n - x^{n-3} - 2x^{n-4} - x^{n-9}$	1.375394454...
10	$x^n - x^{n-3} - 2x^{n-4}$	1.353209964...
11	$x^n - x^{n-3} - 2x^{n-4}$	1.353209964...
12	$x^n - 2x^{n-3} - x^{n-5}$	1.363964602...
13	$x^n - 2x^{n-3} - x^{n-5} - x^{n-9}$	1.385877646...
14	$x^n - 2x^{n-3} - x^{n-6} - x^{n-7}$	1.383834352...

1.33997599527 . . .

This example gives us the bound $m_n = O((1.385877646\ldots)^n)$.

Continuing this process by pulling back 25 levels for $n = 33$, we see that $1.33997599527\ldots$ is the largest real root of the polynomial

$$X^{33} - 752X^8 - 520X^7 - 319X^6 - 231x^5 - 141X^4 - 101X^3 - 54X^2 - 50X - 83,$$

Therefore $m_n = O((1.33997599527\ldots)^n)$.

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Conclusion

Upper Bound

The maximums satisfy $m_n = O((1.33997599527\ldots)^n)$.

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Further Questions

1. Can our bound improvement algorithm be pushed to further lower the bound given more computational power?
2. Is it possible to conclusively prove our conjecture?
3. Furthermore, is there an explicit formula to describe m_n for any arbitrary level?
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Thank you for listening!