Efficient Greek Estimation for Variable Annuities using Monte Carlo Simulation

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The derivative estimate is essential to the hedging of variable annuities. Although resimulation estimate method is widely used, it still has the drawbacks of the bias of estimators and huge computational work. An alternative method which named as pathwise estimate, is explored and discussed in this essay. Besides ordinary "Greeks", for example, delta and rho, this essay also investigates the estimate of derivatives with respect to annual withdrawal amount of guaranteed minimum withdrawal benefit. Apart from traditional resimulation estimate, this essay also analyzes pathwise estimate method with the support of automatic differentiation theory. Based on theoretical analysis and numerical results, this essay shows the advantages of pathwise estimate over resimulation method.

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Chapter 1

Introduction

Hedging strategies should be taken into consideration when writing variable annuity products, and reliable estimates of sensitivities are essential for such strategies, both practically and theoretically (Broadie and Glasserman, 1996). The traditional resimulation estimate method is popular around practitioners, however, it still has several disadvantages, including heavy computational burden and the bias of estimators. This prompted researchers to find a supplement or an alternative to the resimulation estimate. This essay explores the performance of one of the alternatives, which is the pathwise method, in estimating sensitivities of variable annuities, and finds its advantages over the resimulation estimate. Apart from delta and rho, this essay also explores the estimate of derivatives with respect to annual withdrawal amount, based on forward-mode automatic differentiation.

There is a growing literature regarding the pricing and sensitivity estimate of variable annuities. Bacinello et al. (2011) propose a unifying framework for the valuation of variable annuities. Bauer et al. (2008) discuss the pricing of variable annuity contracts under optimal policyholder behavior. The work of Gan (2013) introduces the details of the pricing of variable annuities with the guaranteed minimum withdrawal benefit (GMWB) and the guaranteed minimum death benefit (GMDB) using Monte Carlo simulation. Broadie and Glasserman (1996) investigates the application of the pathwise method and likelihood ratio method in the estimate of security price derivatives, and Broadie and Kaya (2004) includes more details about simulation algorithm and numerical examples on that topic. Cathcart et al. (2015) extend the sensitivity estimate from securities to variable annuities, and Giles and Glasserman (2005) gives an important conclusion that the computational complexity of the adjoint calculation is no more than 4 times greater than the complexity of the original algorithm. Zhou (2017) presents a detailed explanation of the reverse-mode automatic differentiation.

The essay is organized as follows: In Chapter 2, we provide an overview of variable annuities, including the definition of GMDB and GMWB. A Monte Carlo simulation model for pricing variable annuities with GMWB and GMDB riders is also presented in Chapter 2. In Chapter 3, an introduction of the resimulation estimate and pathwise estimate is presented, and it also introduces the details of the pathwise estimate of derivatives with respect to annual withdrawal amount based on the idea of automatic differentiation. Chapter 4 demonstrated numerical results of derivative estimates, as well as the analysis based on these outcomes. In Chapter 5, the conclusion of the research is made, and future research directions are also presented.

Chapter 2

Review of Variable Annuities

In this chapter, we give a brief introduction of variable annuities, and describe the most common guarantees of them. A general pricing approach of variable annuities via Monte Carlo simulation is also provided in this chapter.

2.1 Basics of Variable Annuities

A variable annuity is a type of annuity contract that allows for the accumulation of capital on a tax-deferred basis, designed for obtaining a post-retirement income (Bauer et al., 2008). Basically, a variable annuity can be considered as a fund-linked insurance contract, including a package of financial options on the policy account value (Bauer et al., 2008). Compared with traditional life insurance products, the main feature of variable annuities is that they are associated with a large variety of possible guarantees.

Guarantees, also known as guaranteed minimum benefits, can be divided into two types: guaranteed minimum death benefits(GMDB) and guaranteed minimum living benefits(GMLB). The GMLB options can be categorized into three main groups: guaranteed minimum accumulation benefits(GMAB), guaranteed minimum income benefits(GMIB) and guaranteed minimum withdrawal benefits(GMWB) (Bacinello et al., 2011). In this essay, we focus on the pricing estimate and derivative estimate for GMDB and GMWB.

If the insured dies during the deferment period, the beneficiary will obtain a death benefit. The GMDB is usually available during the accumulation period, while some insurers are willing to provide it also after retirement (Bacinello et al., 2011). The GMWB allows periodical withdrawal from the policy account, even if the account value reduces to zero.

At maturity, the policyholder can take out or annuitize any remaining funds if the account value did not vanish due to withdrawals. In this case, since the policyholders are permitted to withdraw money from the account when investments performances are unsatisfactory, it reduces the risk which policyholders have to bear.

2.2 Pricing GMWB and GMDB via Monte Carlo

The work of Gan (2013) gives notations and formulas for the pricing of variable annuities.

Table 2.1: Notations of Variable Annuity Pricing

Symbol	Meaning
S_t	The underlying mutual fund at time t of the variable annuity
A_t	The account value at time t
W_t	The withdrawal benefit at time t
D_t	The death benefit at time t
G_t^W	The remaining total amount that can be withdrawn after time t
G_t^E	The maximum amount that can be withdrawn annually
G_t^D	The guaranteed minimum death benefit at time t
x_W	The proportion of the premium that can be withdrawn annually
T	The maturity of the contract

Also, in order to distinguish between the values of a state variable (e.g., A_t) immediately before and after the occurrence of such event, we use $(.)_t^-$ and $(.)_t^+$ to denote the two values, respectively.

In our Monte Carlo valuation of the variable annuity contracts, we have those assumptions:

- the underlying mutual fund is simulated as:
  $S_0 = 1$, $S_t = S_{t-1} \cdot \exp\left(\left[r - \frac{1}{2}\sigma^2\right] + \sigma Z\right)$
  for t = 1, 2, ..., T, where r denotes the interest rate, σ denotes the volatility, and Z is a standard normal random variable. In this essay, we assume that $r = 3\%$ and $\sigma = 20\%$, and the number of paths is 1000;

- all the events happen only at anniversary date;

- for a contract with the GMWB rider, the policyholder takes maximum annual withdrawals;
- there are no fees or lapses;
- the mortality follows the 1996 IAM mortality tables provided by the Society of Actuaries

At time $t = 0$, we have

$G_0^W = A_0$, $G_0^E = x_W A_0$, $G_0^D = A_0$.

For $t = 0, 1, ..., T$-1, first we consider the evaluation of the state variables between t^+ and $(t+1)^-$, which is described as follows. The account value evolves as

$$A_{t+1}^- = A_t^+ \frac{S_{t+1}}{S_t}$$

In the time interval between t^+ and $(t+1)^-$, the guaranteed minimum death benefit, the maximum amount can be withdrawn annually, and the remaining total amount that can be withdrawn, do not change, i.e.,

$G_{t+1}^{D-} = G_t^{D+}$, $G_{t+1}^{E-} = G_t^{E+}$, $G_{t+1}^{W-} = G_t^{W+}$

The evolution of the state variables between $(t+1)^-$ and $(t+1)^+$ is described as follows. The death benefit at time $t+1$ is calculated as

$$D_{t+1} = \max(0, G_{t+1}^{D-} - A_{t+1}^-)$$

Since we have the assumption that the policyholder always takes maximally available withdrawals annually, the withdrawal amount at year $t+1$ is given by

$$E_{t+1} = \min(G_{t+1}^{E-}, G_{t+1}^{W-})$$

and the maximum amount that can be withdrawn annually does not change, i.e., $G_{t+1}^{E+} = G_{t+1}^{E-}$. The withdrawal benefit at time $t+1$ is given by

$$W_{t+1} = \max(0, E_{t+1} - A_{t+1}^-)$$

The account value becomes

$$A_{t+1}^{+} = \max(0, A_{t+1}^{-} - E_{t+1})$$

The remaining total amount that can be withdrawn after time $t+1$ becomes

$$G_{t+1}^{W+} = \max(0, G_{t+1}^{W-} - E_{t+1})$$

The guaranteed minimum death benefit will be adjusted pro rata as follows:

$$G_{t+1}^{D+} = \frac{A_{t+1}^{+}}{A_{t+1}^{-}} G_{t+1}^{D-}$$

Then the present value of the GMDB and the GMWB benefits is given by

$$PV = \sum_{t=1}^{T} {}_{t-1}p_{x_0}(1 - q_{x_0+t-1})W_t e^{-rt} + \sum_{t=1}^{T} {}_{t-1}p_{x_0} q_{x_0+t-1} D_t e^{-rt} \tag{2.1}$$

where x_0 is the age of the policyholder. We treat the value of the GMDB rider and the GMWB rider as the average of PV along all paths.

It can be noticed that the pricing of variable annuities is based on the calculation, storage and reuse of intermediate variables, for instance, A_t^-, A_t^+, G_t^W, G_t^D, and so on. With these intermediate variables, the values of W_t and D_t can be obtained step by step, and finally the present value can be calculated. More importantly, these intermediate variables can be applied in the process of the derivative estimate, which is analyzed in Chapter 3.

Chapter 3

Sensitivities for Variable Annuities

The theoretical background and practical methodology of variable annuity sensitivity estimate are discussed in this chapter. The first section gives a brief introduction of the resimulation method, pathwise method, and automatic differentiation theory. The second section demonstrates the expressions of the pathwise estimate of delta, rho, and derivatives with respect to annual withdrawal amounts.

3.1 Sensitivity Estimation Methods

This section presents a general description of the resimulation estimate and the pathwise estimate, and compares the advantages and disadvantages of the two methods. Automatic differentiation theory is also introduced in this section, since it provides methodological support for the pathwise estimate.

3.1.1 Resimulation estimate

A natural finite difference approach, which is named resimulation method, is popular in practice. The work of Broadie and Glasserman (1996), Cathcart et al. (2015) and Glasserman (2013) give explanations of this method. An initial simulation is run to determine a base price, then the parameter which is taken the derivative with respect to is perturbed, and a second simulation is run to determine a perturbed price. The estimate of the derivative is the difference between these two simulated prices divided by the parameter perturbation.

Suppose that the present value PV of a variable annuity depends on a parameter θ and we want to estimate $\frac{dPV}{d\theta}$ at $\theta = \theta_0$. Denote the simulation estimator of the price at $\theta = \theta_0$ by $PV(\theta_0)$, then the estimated price is the sample average over independent simulations of $PV(\theta_0)$. Then the parameter will be perturbed to $\theta_1 = \theta_0 + h$, and the new simulation estimator $PV(\theta_1)$ can be computed.

Here we need to introduce the concept of finite difference. A finite difference is a mathematical expression of the form $f(x+b) - f(x+a)$ (Wilmott et al., 1995). It has three types: forward, backward, and central finite differences. Generally, the first-order forward, backward, and central differences can be defined as follows:

- forward finite difference: $\triangle_h f(x) = f(x+h) - f(x)$
- backward finite difference: $\bigtriangledown_h f(x) = f(x) - f(x-h)$
- central finite difference: $\delta_h f(x) = f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)$

where h is a constant (Petro, 2016).

The resimulation estimator of the derivative is the forward finite difference divided by h, which is $\frac{PV(\theta_1) - PV(\theta_0)}{h}$. In order to meet the definition of a sensitivity, the value of h should be relatively small. Similarly, the resimulation estimate is the average over all trials of this estimator. A common set of random numbers should be used when calculating the two simulation estimators.

From the description of the resimulation estimate, we can see that one of the advantages of it is that it involves no programming effort beyond what is required for the pricing simulation itself (Broadie and Glasserman, 1996). However, the outcome of the resimulation estimate is biased resulting from finite difference approximation to the derivative. Also, it can be inferred that estimating finite differences with respect to n parameters requires $n+1$ simulations, which brings considerable computational burden.

3.1.2 Pathwise estimate

Regarding these disadvantages of the resimulation method, new methods of estimating derivatives were developed, including the pathwise estimate, the likelihood ratio estimate, and a mixture of these two methods. In this essay, we focus on the pathwise method, which avoids simulating at multiple parameter values. Under appropriate conditions, unbiased estimators can be calculated using this method. The work of Broadie and Glasserman (1996) uses the pathwise estimate to investigate derivatives of three types of products, including a path independent model (European options), a path dependent model (Asian options), as well as a model with multiple state variables (options with stochastic volatility). Cathcart et al. (2015) extends the application of the pathwise method and the likelihood ratio method to the sensitivity estimate of GMWB, of which the calculation process includes more intermediate variables and hence more complicated compared with Asian options.

Compared with the resimulation estimate, the pathwise method has several advantages:

1. It can provide unbiased estimate under appropriate conditions. Let $PV(\theta)$ denote the discounted payoff of a variable annuity, then the present value of the variable annuity is given by $\alpha(\theta) = E[PV(\theta)]$. Consider the estimator with respect to θ, we can find the derivative of $\alpha(\theta) = E[PV(\theta)]$ analytically along each simulation path using

$$PV^{'}(\theta) = \lim_{h \to 0} \frac{PV(\theta + h) - PV(\theta)}{h}$$

If the interchanging of differentiation and taking expectations is justified, that is if

$$E\left[\frac{d}{d\theta} PV(\theta)\right] = \frac{d}{d\theta} E[PV(\theta)]$$

then $\frac{1}{n}\sum_{i=1}^{n} Y_i^{'}(\theta)$ is an unbiased estimator of $\alpha^{'}(\theta)$ (Cathcart et al., 2015) (Broadie and Glasserman, 1996).

2. Even more important is the fact that the computational savings. All n derivatives can be estimated from a single simulation, which avoids the repeat of pricing simulation and brings improvement in speed. We will show this point more intuitively in later discussion.

3.1.3 Automatic differentiation

In this essay, apart from Greeks that are commonly seen in computational finance, i.e., derivatives with respect to initial account value (delta) and interest rate (rho), the estimate of derivatives with respect to annual withdrawal amount is also studied. In virtue of the complicated nature of variable annuity pricing, automatic differentiation is considered to be an appropriate approach to implement the pathwise estimate.

Automatic differentiation, also called algorithmic differentiation or computational differentiation, is a set of techniques to numerically evaluate the derivative of a function specified by a computer program. By applying chain rule, derivatives of arbitrary order can be computed efficiently. Compared with classical symbolic differentiation and numerical differentiation, automatic differentiation solves the problem of coding inefficiency and round-off errors, although it has a higher requirement on software (Neidinger, 2010).

Automatic differentiation has two different modes. In forward-mode auto differentiation, we first fix the independent variable to which differentiation is performed and computes the derivative of each sub-expression recursively, i.e.,

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial w_1}\frac{\partial w_1}{\partial x} = \frac{\partial y}{\partial w_1}\left(\frac{\partial w_1}{\partial w_2}\frac{\partial w_2}{\partial x}\right) = \frac{\partial y}{\partial w_1}\left(\frac{\partial w_1}{\partial w_2}\left(\frac{\partial w_2}{\partial w_3}\frac{\partial w_3}{\partial x}\right)\right) = \cdots$$

In reverse-mode automatic differentiation, we first fix the dependent variable to be differentiated and computes the derivative with respect to each sub-expression recursively, i.e.,

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial w_1}\frac{\partial w_1}{\partial x} = \left(\frac{\partial y}{\partial w_1}\frac{\partial w_1}{\partial w_2}\right)\frac{\partial w_2}{\partial x} = \left(\left(\frac{\partial y}{\partial w_1}\frac{\partial w_1}{\partial w_2}\right)\frac{\partial w_2}{\partial w_3}\right)\frac{\partial w_3}{\partial x} = \cdots$$

The derivative with respect to annual withdrawal amount estimate in this paper makes the use of the forward-mode auto differentiation. As mentioned above, the estimate of derivatives can be achieved by adding some extra calculations in the pricing estimate, which will be explained in detail in next section.

3.2 Pathwise Estimate of Derivatives

Since the process of the resimulation estimate, which is introduced in the previous section, is almost same among the different derivatives, in this part we concentrate on the details of the pathwise estimate of derivatives. First, we will discuss the estimate of two "ordinary" Greeks, including delta and rho, then the case of annual withdrawal amount.

3.2.1 Estimate of delta

The derivative delta is defined as the sensitivity of the pay-off with respect to initial account value. Applying chain rule, it can be derived from 2.1 that

$$\text{delta} = \sum_{t=1}^{T}\frac{\partial PV}{\partial D_t}\cdot\frac{\partial D_t}{\partial A_0} + \sum_{t=1}^{T}\frac{\partial PV}{\partial W_t}\cdot\frac{\partial W_t}{\partial A_0}$$

where

$$\frac{\partial PV}{\partial W_t} = {}_{t-1}p_{x_0}(1 - q_{x_0+t-1})e^{-rt}$$
$$\frac{\partial PV}{\partial D_t} = {}_{t-1}p_{x_0}q_{x_0+t-1}e^{-rt}$$

Since $D_{t+1} = \max(0, G_{t+1}^{D-} - A_{t+1}^{-})$,

$$\frac{\partial D_t}{\partial A_0} = \frac{\partial D_t}{\partial(G_t^{D-} - A_t^{-})} \cdot \left(\frac{\partial G_t^{D-}}{\partial A_0} - \frac{\partial A_t^{-}}{\partial A_0} \right)$$

with $\frac{\partial D_t}{\partial(G_t^{D-} - A_t^{-})} = 1_{\{G_t^{D-} > A_t^{-}\}}$.

Since

$$\begin{aligned} G_t^{D-} &= G_{t-1}^{D+} = \frac{A_{t-1}^{+}}{A_{t-1}^{-}} G_{t-1}^{D-} = \prod_{k=1}^{t-1} \frac{A_k^{+}}{A_k^{-}} \cdot A_0 = A_{t-1}^{+} \cdot \prod_{k=0}^{t-2} \frac{A_k^{+}}{A_{k+1}^{-}} \\ &= A_{t-1}^{+} \cdot \prod_{k=0}^{t-2} \frac{S_k}{S_{k+1}} \\ &= A_{t-1}^{+} \cdot \frac{1}{\left[\exp\left(r - \frac{1}{2}\sigma^2 + \sigma Z\right)\right]^{t-1}} \end{aligned}$$

it is clear that $\frac{\partial G_t^{D-}}{\partial A_0} = 0$.

Also we have $\frac{\partial A_t^{-}}{\partial A_0} = \prod_{k=1}^{t-1} \frac{\partial A_k^{+}}{\partial A_k^{-}} \cdot \prod_{k=1}^{t-1} \frac{\partial A_{k+1}^{-}}{\partial A_k^{+}} \cdot \frac{\partial A_1^{-}}{\partial A_0}$

Since $A_t^{-} = A_{t-1}^{+} \cdot \frac{S_t}{S_{t-1}}$, $A_t^{+} = \max(0, A_t^{-} - E_t)$, we have

$$\begin{aligned} \frac{\partial A_t^{-}}{\partial A_{t-1}^{+}} &= \frac{S_t}{S_{t-1}} \\ \frac{\partial A_t^{+}}{\partial A_t^{-}} &= 1_{\{A_t^{-} > E_t\}} \end{aligned}$$

So

$$\begin{aligned}\frac{\partial D_t}{\partial A_0} &= \frac{\partial D_t}{\partial (G_t^{D-} - A_t^-)} \cdot \left(\frac{\partial G_t^{D-}}{\partial A_0} - \frac{\partial A_t^-}{\partial A_0} \right) \\ &= 1_{\{G_t^{D-} > A_t^-\}} \cdot \left(0 - \frac{S_t}{S_0} \cdot \prod_{k=1}^{t-1} 1_{\{A_k^- > E_k\}} \right)\end{aligned}$$

As for $\frac{\partial W_t}{\partial A_0}$, since $W_t = \max(0, E_t - A_t^-)$,

$$\frac{\partial W_t}{\partial A_0} = \frac{\partial W_t}{\partial (E_t - A_t^-)} \cdot \left(\frac{\partial E_t}{\partial A_0} - \frac{\partial A_t^-}{\partial A_0} \right)$$

where $\frac{\partial W_t}{\partial (E_t - A_t^-)} = 1_{\{E_t > A_t^-\}}$.

Since $E_t = \min(G_t^{E-}, G_t^{W-})$, $G_{t+1}^{E-} = G_t^{E+}$, $G_{t+1}^{W-} = G_t^{W+}$, $G_0^E = x_W A_0$, $G_0^D = A_0$, we can obtain that

$$\frac{\partial E_t}{\partial A_0} = \begin{cases} x_W & 0 < t \leqslant \frac{1}{x_W} \\ 1 - (t-1)x_W & \frac{1}{x_W} < t \leqslant \frac{1}{x_W} + 1 \\ 0 & t > \frac{1}{x_W} + 1 \end{cases}$$

and $\frac{\partial A_t^-}{\partial A_0} = \frac{S_t}{S_0} \cdot \prod_{k=1}^{t-1} 1_{\{A_k^- > E_k\}}$ same as above.

Similarly, the derivative with respect to interest rate r, which is named rho, can be obtained using this method. Also, this method can be generalized to the case of higher-order derivatives.

3.2.2 Estimate of rho

The derivative rho is defined as the sensitivity of the pay-off with respect to the interest rate. Similarly, it can also be calculated by chain rule, as

$$\text{rho} = \sum_{t=1}^{T} \frac{\partial PV}{\partial D_t} \cdot \frac{\partial D_t}{\partial r} + \sum_{t=1}^{T} \frac{\partial PV}{\partial W_t} \cdot \frac{\partial W_t}{\partial r}$$

where $\frac{\partial PV}{\partial D_t}$ and $\frac{\partial PV}{\partial W_t}$ given in the discussion of delta above.

Since $D_t = \max(0, G_t^{D-} - A_t^-)$,

$$\frac{\partial D_t}{\partial r} = \frac{\partial D_t}{\partial G_t^{D-}} \cdot \frac{\partial G_t^{D-}}{\partial r} + \frac{\partial D_t}{\partial A_t^-} \cdot \frac{\partial A_t^-}{\partial r}$$

where $\frac{\partial D_t}{\partial G_t^{D-}} = 1_{\{G_t^{D-} > A_t^-\}}$, $\frac{\partial D_t}{\partial A_t^-} = -1_{\{G_t^{D-} > A_t^-\}}$.

Also we know that, $G_t^{D-} = G_{t-1}^{D+} = \frac{A_{t-1}^+}{A_{t-1}^-} G_{t-1}^{D-}$, so

$$\frac{\partial G_t^{D-}}{\partial r} = \frac{\partial G_{t-1}^{D+}}{\partial r} = \frac{\partial G_{t-1}^{D+}}{\partial A_{t-1}^+} \cdot \frac{\partial A_{t-1}^+}{\partial r} + \frac{\partial G_{t-1}^{D+}}{\partial A_{t-1}^-} \cdot \frac{\partial A_{t-1}^-}{\partial r} + \frac{\partial G_{t-1}^{D+}}{\partial G_{t-2}^{D+}} \cdot \frac{\partial G_{t-2}^{D+}}{\partial r}$$

where $\frac{\partial G_{t-1}^{D+}}{\partial A_{t-1}^+} = \frac{G_{t-1}^{D-}}{A_{t-1}^-}$, $\frac{\partial G_{t-1}^{D+}}{\partial A_{t-1}^-} = -\frac{A_{t-1}^+ \cdot G_{t-1}^{D-}}{(A_{t-1}^-)^2}$, $\frac{\partial G_{t-1}^{D+}}{\partial G_{t-2}^{D+}} = \frac{A_{t-1}^+}{A_{t-1}^-}$.

Also we have

$$\begin{aligned} \frac{\partial A_{t-1}^+}{\partial r} &= \prod_{k=1}^{t-1} \frac{\partial A_k^+}{\partial A_k^-} \cdot \prod_{k=1}^{t-2} \frac{\partial A_{k+1}^-}{\partial A_k^+} \cdot \frac{\partial A_1^-}{\partial r} \\ &= \prod_{k=1}^{t-1} 1_{\{A_k^- > E_k\}} \cdot \frac{S_{t-1}}{S_1} \cdot \frac{\partial A_1^-}{\partial r} \end{aligned}$$

Since $A_1^- = A_0 \cdot \frac{S_1}{S_0} = A_0 \exp\left(r - \frac{1}{2}\sigma^2 + \sigma Z\right)$,

$$\frac{\partial A_{t-1}^+}{\partial r} = \prod_{k=1}^{t-1} 1_{\{A_k^- > E_k\}} \cdot \frac{S_{t-1}}{S_1} \cdot A_0 \exp\left(r - \frac{1}{2}\sigma^2 + \sigma Z\right)$$

Similarly,

$$\begin{aligned} \frac{\partial A_{t-1}^-}{\partial r} &= \prod_{k=1}^{t-2} \frac{\partial A_k^+}{\partial A_k^-} \cdot \prod_{k=1}^{t-2} \frac{\partial A_{k+1}^-}{\partial A_k^+} \cdot \frac{\partial A_1^-}{\partial r} \\ &= \prod_{k=1}^{t-2} 1_{\{A_k^- > E_k\}} \cdot \frac{S_{t-1}}{S_1} \cdot A_0 \exp\left(r - \frac{1}{2}\sigma^2 + \sigma Z\right) \end{aligned}$$

As for $\frac{\partial G_{t-2}^{D+}}{\partial r}$ and $\frac{\partial A_t^-}{\partial r}$, we can see that it has the same form with $\frac{\partial G_{t-1}^{D+}}{\partial r}$ and $\frac{\partial A_{t-1}^-}{\partial r}$ respectively, which is discussed in detail above. Also, it suggests that during the process of calculating derivatives using the pathwise method, intermediate variables (not only $\frac{\partial G_{t-1}^{D+}}{\partial r}$) can be stored and reused repeatedly, which brings many computational savings.

3.2.3 Estimate of derivatives with respect to annual withdrawal amount

Now we want to investigate how annual withdrawal amount affects the present value of variable annuities. More precisely, we want to see the effects on present value from withdrawals occurring at different years. From the analysis in the previous chapter, it is clear that once the premium, the withdrawal rate and the maturity are determined, the annual withdrawals amounts are determined as well. In other words, we can calculate annual withdrawals amounts using the three values, and treat these amounts as a vector, which can be an input when calculating the present value.

From the analysis in the previous chapter, the pricing formula of a variable annuity is defined as follows:

$$PV = \sum_{t=1}^{T} {}_{t-1}p_{x_0}(1 - q_{x_0+t-1})W_t e^{-rt} + \sum_{t=1}^{T} {}_{t-1}p_{x_0} q_{x_0+t-1} D_t e^{-rt}$$

$$\frac{\partial PV}{\partial E_s} = \sum_{t=1}^{T} \frac{\partial PV}{\partial W_t} \cdot \frac{\partial W_t}{\partial E_s} + \sum_{t=1}^{T} \frac{\partial PV}{\partial D_t} \cdot \frac{\partial D_t}{\partial E_s}\ , \text{ for } s = 1, 2, \cdots, T,\ t = s, \cdots, T$$

where $\frac{\partial PV}{\partial W_t} = {}_{t-1}p_{x_0}(1 - q_{x_0+t-1})e^{-rt}$, $\frac{\partial PV}{\partial D_t} = {}_{t-1}p_{x_0} q_{x_0+t-1} e^{-rt}$.

First, we consider the $\frac{\partial PV}{\partial W_t}$. Since $W_t = \max(0, E_t - A_t^-)$,

When $t = s$, $\frac{\partial W_t}{\partial E_s} = \frac{\partial W_t}{\partial E_t} = 1_{\{E_t > A_t^-\}}$.

When $t = s + 1, \cdots, T$,

$$\frac{\partial W_t}{\partial E_s} = \frac{\partial W_t}{\partial A_t^-} \cdot \frac{\partial A_t^-}{\partial A_{t-1}^+} \cdot \left[\prod_{k=s}^{t-2} \frac{\partial A_{k+1}^+}{\partial A_{k+1}^-} \cdot \frac{\partial A_{k+1}^-}{\partial A_k^+} \right] \cdot \frac{\partial A_s^+}{\partial E_s}$$

From the analysis above, the relevant derivatives can be calculated as follows:

$$\begin{aligned}
\frac{\partial W_t}{\partial A_t^-} &= -1_{\{E_t > A_t^-\}} \\
\frac{\partial A_t^-}{\partial A_{t-1}^+} &= \frac{S_t}{S_{t-1}} \\
\frac{\partial A_t^+}{\partial A_t^-} &= 1_{\{A_t^- > E_t\}} \\
\frac{\partial A_t^+}{\partial E_t} &= -1_{\{A_t^- > E_t\}}
\end{aligned}$$

It is clear that the calculation of those derivatives can be embedded to the process of pricing of variable annuities. Also, $\left[\prod_{k=s}^{t-2} \frac{\partial A_{k+1}^+}{\partial A_{k+1}^-} \cdot \frac{\partial A_{k+1}^-}{\partial A_k^+}\right] \cdot \frac{\partial A_s^+}{\partial E_s}$ can be treated as $\frac{\partial A_{t-1}^+}{\partial E_s}$, which will be reused in the later calculation.

Regarding the calculation of $\frac{\partial D_t}{\partial E_s}$, since $D_t = \max(0, G_t^{D-} - A_t^-)$,

$$\frac{\partial D_t}{\partial E_s} = \frac{\partial D_t}{\partial (G_t^{D-} - A_t^-)} \cdot \left(\frac{\partial G_t^{D-}}{\partial E_s} - \frac{\partial A_t^-}{\partial E_s}\right) \text{ , for } s = 1, 2, \cdots, T, \; t = s+1, \cdots, T$$

where $\frac{\partial D_t}{\partial (G_t^{D-} - A_t^-)} = 1_{\{G_t^{D-} > A_t^-\}}$.

From the analysis above,

$$\begin{aligned}
G_t^{D-} &= G_{t-1}^{D+} \\
&= \frac{A_{t-1}^+}{A_{t-1}^-} \cdot G_{t-1}^{D-} \\
&= \frac{A_{t-1}^+}{A_{t-1}^-} \cdot G_{t-2}^{D+} \\
&= \frac{A_{t-1}^+}{A_{t-1}^-} \cdot \frac{A_{t-2}^+}{A_{t-2}^-} \cdot G_{t-3}^{D+} \\
&= \frac{A_{t-1}^+}{A_{t-1}^-} \cdot \frac{A_{t-2}^+}{A_{t-2}^-} \cdots \frac{A_s^+}{A_s^-} \cdot G_s^{D-}
\end{aligned}$$

Since $A_{t-1}^{-} = A_{t-2}^{+} \cdot \frac{S_{t-1}}{S_{t-2}}$,

$$\begin{aligned} G_{t-1}^{D+} &= A_{t-1}^{+} \cdot \frac{S_{t-2}}{S_{t-1}} \cdot \frac{S_{t-3}}{S_{t-2}} \cdots \frac{S_s}{S_{s+1}} \cdot \frac{G_s^{D-}}{A_s^{-}} \\ &= A_{t-1}^{+} \cdot \frac{S_s}{S_{t-1}} \cdot \frac{G_s^{D-}}{A_s^{-}} \end{aligned}$$

where $\frac{S_s}{S_{t-1}} \cdot \frac{G_s^{D-}}{A_s^{-}}$ is a constant with respect to E_t.

Consequently,

$$\frac{\partial G_t^{D-}}{\partial E_s} = \frac{\partial G_{t-1}^{D+}}{\partial A_{t-1}^{+}} \cdot \frac{\partial A_{t-1}^{+}}{\partial E_s} = \frac{S_s}{S_{t-1}} \cdot \frac{G_s^{D-}}{A_s^{-}} \cdot \frac{\partial A_{t-1}^{+}}{\partial E_s}$$

where $\frac{\partial A_{t-1}^{+}}{\partial E_s}$ is discussed above.

As for $\frac{\partial A_t^{-}}{\partial E_s}$, similarly,

$$\begin{aligned} \frac{\partial A_t^{-}}{\partial E_s} &= \frac{\partial A_t^{-}}{\partial A_{t-1}^{+}} \cdot \frac{\partial A_{t-1}^{+}}{\partial E_s} \\ &= \frac{S_t}{S_{t-1}} \cdot \frac{\partial A_{t-1}^{+}}{\partial E_s} \end{aligned}$$

Based on the analysis above,

$$\frac{\partial D_t}{\partial E_s} = 1_{\{G_t^{D-} > A_t^{-}\}} \cdot \left(\frac{S_s}{S_{t-1}} \cdot \frac{G_s^{D-}}{A_s^{-}} - \frac{S_t}{S_{t-1}} \right) \cdot \frac{\partial A_{t-1}^{+}}{\partial E_s} \text{ , for } s = 1, 2, \cdots, T, \ t = s+1, \cdots, T$$

where $\frac{S_s}{S_{t-1}} \cdot \frac{G_s^{D-}}{A_s^{-}}$ is a constant with respect to E_s.

Based on the analysis above, we can notice that a vital property of pathwise estimate is the application of intermediate variables. In the traditional resimulation method, when estimating n derivatives, the number of simulations required is $n+1$, which means $n+1$ times of computational efforts. However, using the pathwise method, with the application of intermediate variables, the derivative estimate can be finished in only one valuation.

The new intermediate variables, which are needed for the estimate of derivatives (for instance, $\frac{\partial A_{t-1}^{+}}{\partial E_s}$), are calculated with those original intermediate variables calculated in the pricing process, such as A_t^{-}, A_t^{+}, G_t^{W}, G_t^{D}, etc. This means that the calculation of new intermediate variables can be implemented right after the calculation of original ones. In the view of coding, this just brings a few lines of codes added on the initial pricing codes. Associated with the unbiased property of pathwise estimators, which is introduced in the previous section, we can assume that pathwise method enhances the efficiency of the derivative estimate, and can be considered as a satisfying alternative of the resimulation method. Some numerical results are presented in Chapter 4.

Chapter 4

Numerical Results

In this part, we present several numerical results of the resimulation and the pathwise estimate of variable annuities sensitivities, including delta, rho and derivatives with respect to annual withdrawal amount. We assume that the account value $A_0 = 10^5$, the age of the insured is 50 and the gender is female, the maturity is 10, 20, 25, respectively. The reason why these three number of maturities are chosen is that they cover the cases of short, medium and long maturity, which can give a better view of the effect of annual withdrawal amount on present value. Also, we let withdrawal rate x_W equal to 1/maturity, which means that the withdrawal rate is 10%, 5%, 4% correspondently. Under this assumption of withdrawal rate, the annual withdrawal amount of each year is equal to $x_W A_0$, which is designed for computational and analytical convenience.

Regarding the choice of h of the resimulation methods, it is worth noting that there is a trade-off on the choice of h – the smaller h can decrease the bias in the estimate, however, it will increase the variance of the estimator at the same time (Bacinello et al., 2011). We choose $h = 1$ for the delta resimulation and $h = 10^{-6}$ for the rho resimulation. When calculating the sensitivities with respect to annual withdrawal amount, we set h equals to 1% of annual withdrawal amount. In the case of rho estimate, since the value of h is relatively small, the value of rho in the table is the original estimate value divided by 10^4 for analytical convenience.

Table 4.1: Resimulation and pathwise estimate of derivatives of variable annuities

	Maturity					
	10	(Std.Err)	20	(Std.Err)	25	(Std.Err)
Present Value						
Simulation estimate	8037.865	365.896	7260.205	326.623	6567.016	298.318
Delta						
Resimulation estimate	-0.243	0.009	-0.175	0.007	-0.147	0.006
Pathwise estimate	-0.245	0.009	-0.178	0.007	-0.151	0.006
Rho						
Resimulation estimate	-18.159	0.645	-26.086	0.928	-27.085	0.982
Pathwise estimate	-18.159	0.645	-26.086	0.928	-27.085	0.982
E_1						
Resimulation estimate	0.271	0.011	0.191	0.008	0.157	0.007
Pathwise estimate	0.269	0.011	0.191	0.008	0.157	0.007
E_T						
Resimulation estimate	0.329	0.011	0.229	0.008	0.181	0.006
Pathwise estimate	0.327	0.011	0.229	0.008	0.181	0.006

Parameters: r = 3%, σ = 20%. All simulation results based on 1000 trials.

Here we treat the resimulation method as a "benchmark" to test the performance of the pathwise method. First, from the table, it can be noticed that both of the point estimates and standard errors of the resimulation and pathwise methods are very similar, especially in the case with longer maturity, which means that the pathwise estimate is acceptable. The reason why the outcomes of two methods are not identical can be explained as follows.

Take the example of the derivative with respect to E_T, from the analysis above, the pathwise estimator of $\frac{\partial PV}{\partial E_T}$ is

$$\frac{\partial PV}{\partial E_T} = \frac{\partial PV}{\partial W_T} \cdot \frac{\partial W_T}{\partial E_T} + \frac{\partial PV}{\partial D_T} \cdot \frac{\partial D_T}{\partial E_T}$$

Since $D_T = \max(0, G_T^{D-} - A_T^-)$, $G_T^{D-} = G_{T-1}^{D+}$, $A_T^- = A_{T-2}^+ \cdot \frac{S_{T-1}}{S_{T-2}}$, $\frac{\partial D_T}{\partial E_T} = 0$. Then in the case of pathwise estimator, $\frac{\partial PV}{\partial E_T} = \frac{\partial PV}{\partial W_T} \cdot \frac{\partial W_T}{\partial E_T} = {}_{T-1}p_{x_0}(1 - q_{x_0+T-1})e^{-rT} \cdot 1_{\{E_T > A_T^-\}}$.

Regarding the resimulation estimator, we use $\{\cdot\}'$ to denote the values after perturbation, and the pathwise estimator is obtained as

$$\frac{\partial PV}{\partial E_T} = \frac{PV' - PV}{h} = \frac{{}_{T-1}p_{x_0}(1 - q_{x_0+T-1})e^{-rT} \cdot (W_T' - W_T)}{h}$$

Since $W_T = \max(0, E_T - A_T^-)$, $W_T' = \max(0, E_T + h - A_T^-)$, the details of the resimulation and pathwise estimator is demonstrated in the table:

Table 4.2: Resimulation and pathwise estimate of the derivative with respect to E_T

	$E_T > A_T^-$	$E_T = A_T^-$
W_T	$E_T - A_T^-$	0
W_T'	$E_T + h - A_T^-$	h
Resimulation estimator	${}_{T-1}p_{x_0}(1 - q_{x_0+T-1})e^{-rT}$	${}_{T-1}p_{x_0}(1 - q_{x_0+T-1})e^{-rT}$
Pathwise estimator	${}_{T-1}p_{x_0}(1 - q_{x_0+T-1})e^{-rT}$	0

	$E_T < A_T^-$		
	$E_T + h > A_T^-$	$E_T + h = A_T^-$	$E_T + h < A_T^-$
W_T	0	0	0
W_T'	$E_T + h - A_T^-$	0	0
Resimulation estimator	${}_{T-1}p_{x_0}(1 - q_{x_0+T-1})e^{-rT} \cdot \left(1 + \frac{E_T - A_T^-}{h}\right)$	0	0
Pathwise estimator	0	0	0

From the table, it is clear that in the paths with "$E_T = A_T^-$" or "$E_T < A_T^-$ and $E_T + h > A_T^-$", the resimulation and pathwise estimator are different, which contributes to the minor difference between the two estimates. The similar problem can appear in the estimate of other derivatives. However, the probability that this discrepancy appears is very small. First, since the account value A_t^- is calculated based on the random variation of underlying stock price, it is almost impossible that the account value at time period t is exactly the same with annual withdrawal amount; Also, regarding the case of "$E_T < A_T^-$ and $E_T + h > A_T^-$" , it is mentioned in the previous discussion that the value of perturbation h is relatively small, hence the probability that "$E_T < A_T^-$ and $E_T + h > A_T^-$" is also small. This is why the two estimates are not identical, but still very close.

The other reason for the divergence of the two estimates is the difference between is the choice of h. To find an "appropriately small" h is the key of the resimulation method. From the analysis above, it can be seen that if h is not small enough, $E_T - A_T^-$ and $E_T + h - A_T^-$ can have different signs, which leads to the estimate error. However, if h is too small, the jump between PV and PV' will not be obvious enough, and the implication of "derivative" can not be fully reflected. This is also one of the problems of the resimulation method, which needs sufficient consideration in the resimulation estimate.

Also, it can be noticed that the derivative with respect to E_T is larger than E_1, which means that the increase in the final annual withdrawal amount will bring higher increase than first withdrawal in present value.

Chapter 5

Conclusion and Future Work

In this essay, we use two different methods, the resimulation estimate and the pathwise estimate, to investigate the derivatives of variable annuities with respect to initial account value A_0 (delta), interest rate r (rho), and annual withdrawal amount E. Also, we did a comparison of the two methods, and find the advantages of the pathwise estimate over the resimulation estimate.

From the analysis above, we can see that the resimulation method is straightforward and requires no extra efforts on coding, however, the multiple valuations bring much computational work, and estimates obtained from the resimulation method are biased. The main advantages of the pathwise method are accuracy, speed, and simpleness. It is able to provide unbiased estimate under appropriate conditions, and by mixing the process of derivatives estimate and pricing estimate, it brings great computational savings. Although it needs extra coding efforts, it can still be considered as a satisfactory estimate method compared with the resimulation method, especially in the case where a large set of derivatives are required. Also, Giles and Glasserman (2005) proved that the computational complexity of the adjoint calculation is no more than 4 times greater than the complexity of the original algorithm, which demonstrates the feasibility of generalizing the pathwise estimate to the cases which require higher-order derivatives.

Future work can be devoted to following issues. First, based on the resimulation estimate and the pathwise estimate presented in this essay, variance reduction techniques can be applied to the original simulation estimators to improve the accuracy of the estimate. Second, the likelihood ratio methods can be employed to the derivative estimate and compared with the pathwise method. Cathcart et al. (2015) mentioned that the combination of the pathwise method and the likelihood ratio method has more satisfying performance when estimating higher-order Greeks, which is relevant to hedging variable annuity exposures since accurate and unbiased estimates of higher-order derivatives allow taking the convexity of liabilities with respect to key market exposures into consideration. Finally, while using forward-mode automatic differentiation to estimate derivative with respect to annual withdrawal amount in this paper, the reverse-mode automatic differentiation is also deserved to explore.

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