The Optimal Strategy in a Semi-static Model for Pricing Guaranteed Minimum Benefit Riders under Different Withdrawal Rate Assumptions

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Contents

Abstract

Pricing of the riders of variable annuity (VA) products is complicated, and it is common practice to assume that the policyholder of a variable annuity (VA) contract with guaranteed minimum withdrawal benefit (GMWB) to withdraw all the time in order to get maximum value out of the riders of the contract. Intuitive as it seems, this assumption does not necessarily represents the optimal strategy all the time. In this paper we use a simplified two-period model to explore the interactive relationship of the values between GMWB and guaranteed minimum death benefit (GMDB). Then we extend our findings to a ten-period model and use crude Monte-Carlo simulation to see whether the findings hold true. We discovered that while it is reasonable to assume that the policyholder always withdraws under the most common withdrawal rate assumptions, it will no longer be optimal if the withdrawal rate is much lower or higher than the inverse of the maturity of the contract.

Keywords: *Variable Annuity, Guaranteed Minimum Withdrawal Benefit, Guaranteed Minimum Death Benefit, Monte-Carlo Simulation, Optimal Strategy*

1 Introduction

1.1 Variable Annuities

Variable annuities are a type of product that has gained increased interest over the past decades. In short, it is an insurance product that allows policyholders to allocate their premiums to investment accounts and therefore gain the corresponding revenue as annuity payouts. Compared to participating insurance contracts, which provides the policyholders dividends according to the performance of the insurance company, variable annuities allow you to choose the allocation of premium in investment funds with different risk level and each policyholder receives different payouts given the performance of the investment fund they choose.

A common feature that comes with variable annuities is the different types of guarantees in such polices. The types of guarantees can be broadly categorized into Guaranteed Minimum Death Benefit(GMDB) and Guaranteed Minimum Living Benefit(GMLB). GMLB can be furthered categorized into Guaranteed Minimum Allocation Benefit(GMAB), Guaranteed Minimum Income Benefit(GMIB) and Guaranteed Minimum Withdrawal Benefit(GMWB). In our paper we will specifically look at GMWB and GMDB riders. Policyholders in contracts with a GMWB rider are allowed to periodically withdraw a specified amount during the accumulation of the annuity until the allowed withdrawal amount is depleted or the contract reaches maturity, regardless of the performance of the investment account. Therefore the GMWB rider protects the policyholders against downside market risks. The GMDB rider allows the policyholder to get a specified payout if he or she expires before maturity. The amount that is guaranteed is usually related to the policyholder's remaining account value and the initial premium.

The complexity of pricing these riders is that as policyholders make withdrawals, the amount of investment and the amount of death benefit will correspondingly decrease. Therefore it is inherently an optimization between the expected value of discounted future cash flows and immediate realized values. This may sound familiar, as we know that the pricing of American call and put options is a fundamentally identical process. Indeed, as in Liu (2010), viewing GMWB as a more dynamic version of the Bermudan option is a strategy widely adopted in pricing VA contracts.

1.2 Pricing of GMWB

To price a GMWB, we need to make some assumptions about the behavior of the policyholders in terms of withdrawal strategy and surrenders. These assumptions decide what type of model we choose. If we assume the policyholders make constant withdrawals and they never surrender their policy, it is a type of **static** model. Once the strategy is fixed, we will be able to track the evolution of the contract value based on the investment-linked fund model we adopt, and price the GMWB by calculating the discounted future cash flows. This was well developed in Milevsky and Salisbury (2005). More complicated versions of this model, as in Bacinello et al. (2011), where they assumed a fixed surrender percentage (which is zero by default), are commonly used to price a GMWB contract. Additional random factors can be added to the model to study the marginal influence of different parameters, Bauer and Ha (2013) assumed stochastic mortality rate or stochastic risk free interest rate.

The most realistic and complicated model for pricing GMWB is the **dynamic** model. In this model, the policyholder can withdraw any amount or choose to surrender the policy at any time before maturity. If the policyholder chooses to withdraw more than the guaranteed amount or surrenders the policy before maturity, he or she will be subject to a penalty fee. This makes the model much more complicated. To utilize this model, Zhuliang Chen (2007) uses partial differential equation, and Anna Rita Bacinello (2013) uses dynamic programming. These methods are computationally more expensive compared to the static pricing model.

A compromise between the dynamic model and the static model, which is often referred to as the **mixed** model, is a model that allows policyholders to surrender at any time in the life of the contract but assumes that the policyholder withdrawals exactly when and what contractually specified . With the choice to surrender the policy, the policyholder must evaluate the continuation value of the policy and compare that with the immediate surrender value. This resembles the pricing of American options, since they are both trying to evaluate the continuation value of the contract at each time step. The challenge is how to decide the continuation value. The most intuitive way is the nested Monte Carlo simulation, which simulates the future value of the contract at each step by attempting to exhaust as many future values as possible and calculate the discounted expected value to decide whether each withdrawal should be made. This method is computationally expensive, if at all possible, even with the modern computing power. Some efforts have been made by Cathcart and Morrison (2009) in pricing the GMWB in short time horizons using nested Monte Carlo. More often, one seeks to reduce the problem of dimension. As proposed by Lonstaff and Shwarts (2001), least-squares Monte Carlo (LSM) is an example of the available tools. Bacinello et al. (2011), Bauer and Ha (2013) and several others has also adopted LSM.

1.3 Motivation of study

We try to explore in this paper whether withdrawing all the time is indeed the optimal strategy in a static model and whether it holds true under different withdrawal rate assumptions. We start by looking at a simplified two-period model and then we try to extend to a more common ten-period model.

The rest of the paper is consisted of two main parts. We first examine the two period model to determine how withdrawal strategies could impact the values of GMWB and GMDB riders. Then we extend our experiment to a ten-period model and test if the optimal strategy is still to withdraw all the time under different withdrawal rates.

2 Model Description

2.1 GMWB, GMDB and the withdrawal strategy

First we need to define the meaning of "value" in our context. GMWB and GMDB riders only generate value when the guaranteed amount is greater than what the insured could have gotten without the riders. For instance, if the insured dies before maturity, GMDB usually guarantees that at least the premium will be returned. This means that if the investment account ends up being more valuable than the guaranteed death benefit, the GMDB rider will eventually have 0 realized value. In terms of GMWB, withdrawing does not generate value by itself, since the insured is practically withdrawing from what he deposited in the investment account in the first place. Value is generated when the investment account is not doing well enough, and the guaranteed withdrawal at a particular term exceeds the investment account value. In this perspective, the function of GMDB and GMWB much resembles that of an option, in that they have positive exercise value when certain conditions are met, and are worthless otherwise.

So how do withdrawal decisions influence the value of the riders? If GMWB is to have value, then withdrawals must be made. But GMDB value on the other hand, is usually negatively correlated to withdrawals. This is because in practice, the insurer usually reduces the guaranteed death benefit after a withdrawal is made. One such adjustment is the pro rata adjustment. This means that after each withdrawal, the GMDB will be reduced by a proportion that is identical or comparable to the proportional reduction in the investment account value that the withdrawal has brought forth.

Therefore intuitively, if one chooses to withdraw at a time when the investment account value drops below a certain level, the withdrawal amount will exceed the investment account value, and the GMWB will have a value equal to the difference between the withdrawal amount and the investment account value at time 1. However, the insured also reduces the GMDB account value in the future since that account is subject to a pro rata adjustment.

2.2 The general model

We adopt a VA contract with GMWB and GMDB riders and use assumptions similar to that of Gan (2013). The assumptions regarding policyholder behaviors for our model are:

- The policyholder can choose to withdraw a specified amount at the end of each year until maturity
- The policyholder never surrenders the policy

At each time step, the policyholder must be in one of the three status:

- The policyholder withdraws the maximum allowable amount
- The policyholder does not withdraw
- The policyholder dies

Compared to Gan (2013), who adopted an entirely static model, we use a semi-static model in which the policyholder can choose whether or not to withdraw at each period. The following are the notations we need.

Notation	Meaning
S_t	Investment-link fund value at time t
A_t	Policyholder's Account value at time t
W_t	Withdrawal benefit at time t
D_t	Death benefit at time t
E_t	Maximum amount that can be withdrawn at time t
K_t	Actual withdrawal amount at time t
A_t^W	Remaining total amount that can be withdrawn at time t
A_t^D	Death benefit account at time t
g	Annual guaranteed withdrawal amount
X_w	Annual guaranteed withdrawal rate
T	Maturity

The initial values are as follows,

$$S_0 = A_0$$
$$A_0^W = A_0, A_0^D = A_0$$
$$g = X_w A_0$$

We use $(.)_t^-$ and $(.)_t^+$ to denote the values immediately before and after the withdrawal decision. For t=0,1... T-1, the account value is correlated to the investment-link fund change

$$A_{t+1}^- = A_t^+ \frac{S_{t+1}}{S_t}$$

The withdrawal account and death benefit account remain unchanged

$$A_{t+1}^{W-} = A_t^{W+}$$
$$A_{t+1}^{D-} = A_t^{D+}$$

Death benefit at time t is

$$D_t = A_t^D$$

At $t = 1, 2, 3...T$, the insured can choose to withdraw g, and the maximum amount that can be withdrawn at time t is

$$E_t = \min(g, A_t^W)$$

Rendering the withdrawal benefit at time t

$$W_t = \max(0, E_t - A_t^-)$$

Depending on whether the policyholder actually withdraws or not at time t, the actual withdraw amount at time t is

$$K_t = \begin{cases} 0, & \text{not withdraw} \\ E_t, & \text{withdraw} \end{cases}$$

The account value after the withdrawal decision is

$$A_t^+ = A_t^- - K_t$$

The evolution of the GMWB and GMDB account is as follows

$$A_t^{W+} = A_t^{W-} - K_t$$
$$A_t^{D+} = A_t^{D-} \frac{A_t^-}{A_t^+}$$

Then the time 0 value of the GMWB and GMDB benefit is

$$\begin{aligned} V_0 = &\sum_{t=1}^{T} {}_{t-1}p_{x_0}(1 - q_{x_0+t-1})W_t e^{-rt} \\ &+ \sum_{t=1}^{T} {}_{t-1}p_{x_0} q_{x_0+t-1} D_t e^{-rt} \end{aligned} \tag{1}$$

2.3 The two-period model

In our experiment, we start by examining the simplest interactive scenario, which is the two-period model, in which the policyholder is allowed to withdraw at time 1 and time 2.

We will study how the value of the riders will be influenced by the withdrawal decision at time 1. We assume that the policyholder can choose to withdraw at time 1 and time 2. However, we will see from the mathematical derivation that, even though at time 2 the insured is allowed to choose to withdraw, the withdrawal does not affect the value of the death benefit.

The evolution of relevant accounts and values are similar to that of the general model. At time 0, we have the initial account value A_0. The initial withdrawal account A_W and death benefit account A_D are both assumed to take the value of A_0 at time 0. A contract specified allowed withdrawal rate for each period is denoted by X_w, and is bounded by $0 \leq X_w \leq 1$. Thus we have the guaranteed withdrawal g at time 1 $g = A_0 X_W$. Given a specific risk free interest rate and a stock volatility σ, we can simulate the random investment account change with R_1 and R_2, the evolution of the account value is now

$$A_1 = A_0 R_1$$

$$A_2 = A_1^+ R_2$$

with A_1^+ representing the account value after the withdrawal is made at time 1.

Now at time 1, assume the policyholder makes the withdrawal if he or she is alive. The GMWB value is denoted by $W_1 = \max(g - A_1, 0)$. The GMDB value at time 1 is denoted by $D_1 = \max(A_D - A_1, 0)$. The new account value after the withdrawal decision becomes

$$A_1^+ = \max(A_1 - g, 0)$$

and the remaining withdrawal account value and death benefit account value becomes

$$A_w^+ = \max(A_w - g, 0)$$

and

$$A_D^+ = A_D \frac{A_1^+}{A_1}$$

At time 2, which is the last period in our scenario, the account value becomes $A_2 = A_1^+ R_2$. The amount that the policyholder can withdraw at time 2 is $\min(A_w^+, g)$. Thus the survival benefit is now the withdrawal minus the account value at time 2, which gives us

$$W_2 = \max\left(\min(A_w^+, g) - A_2, 0\right)$$

Lastly, we also have the GMDB value

$$D_2 = \max(A_D^+ - A_2, 0)$$

With the information above, we have the value of the riders at time 0

$$V_w = p_1 W_1 e^{-r} + q_1 D_1 e^{-r} + p_1 p_2 W_2 e^{-2r} + p_1 q_2 D_2 e^{-2r}$$

with p_1, q_1, p_2, q_2 representing the mortality rates at time 1 at time 2.

Expanding the equation gives

$$\begin{aligned}
V_w = p_1 e^{-r} \max\Big(A_0 X_w - A_0 R_1, 0\Big) \\
+ q_1 e^{-r} \max\Big(A_0 - A_0 R_1, 0\Big) \\
+ p_1 p_2 e^{-2r} \max\Big(\min\big(A_0 X_w, A_0(1 - X_w)\big) \\
- R_2 \max(A_0 R_1 - A_0 X_w, 0), 0\Big) \\
+ p_1 q_2 e^{-2r} \max\Big(A_0 \frac{\max(A_0 R_1 - A_0 X_w, 0)}{A_0 R_1} - (A_0 R_1 - A_0 X_w) R_2, 0\Big)
\end{aligned} \tag{2}$$

If the policyholder chooses not to withdraw at time 1, there would be no GMWB value at time 1, and GMDB value is the same as D_1. At time 2, the policyholder's investment account value becomes $A_2^* = A_0 R_1 R_2$. And the GMWB value and GMDB value at time 2 are

$$W_2^* = \max(g - A_2, 0)$$
$$D_2^* = \max(A_D - A_2, 0) = \max(A_0 - A_2, 0)$$

This gives us the value of both riders at time 0

$$V_w^* = q_1 D_1 e^{-r} + p_1 p_2 W_2^* e^{-2r} + p_1 q_2 D_2^* e^{-2r}$$

Expanding the equation gives

$$\begin{aligned}
V_w^* = q_1 e^{-r} \max(A_0 - A_0 R_1, 0) \\
+ p_1 p_2 e^{-2r} \max(A_0 X_w - A_0 R_1 R_2, 0) \\
+ p_1 q_2 e^{-2r} \max(A_0 - A_0 R_1 R_2, 0)
\end{aligned} \tag{3}$$

By taking the difference of (2) and (3), we get the difference of the combined value of GMWB and GMDB at time 0 between withdrawing and not withdrawing at time 1. Taking out the common parameters gives equation 4

$$
\begin{aligned}
V_w - V_w^* =& (A_0 p_1 e^{-r})\{\max(X_w - R_1, 0) \\
&+ p_2 e^{-r}[\max(X_w - R_2 \max(R_1 - X_w, 0), 0) \\
&- \max(X_w - R_1 R_2, 0), 0)] \\
&+ q_2 e^{-r}[\max(\frac{\max(R_1 - X_w, 0)}{R_1} - (R_1 - X_w) R_2, 0) \\
&- \max(1 - R_1 R_2, 0)]\}
\end{aligned} \tag{4}
$$

3 Numerical Experiment

3.1 The two-period model

We adopt the mortality rate from the 1996IAM mortality table and use age 40 female as a sample policyholder. That gives us $p_1 = 0.999451$ and $p_2 = 0.999407$. Also we set risk-free rate $r = 0.03$ and stock volatility $\sigma = 0.2$. Initial account value $A_0 = 100$ so that the numbers we eventually get are percentages of the original amount. X_w in this case is first set to be 0.5 because the most common contract specification for X_w is $X_w = 1/T$, and in this scenario $T = 2$. Using the log-normal growth factor at time 1 and time 2 (R_1 and R_2) as variables and the time 0 difference $V_w - V_w^*$ as dependent, we get Figure 1.

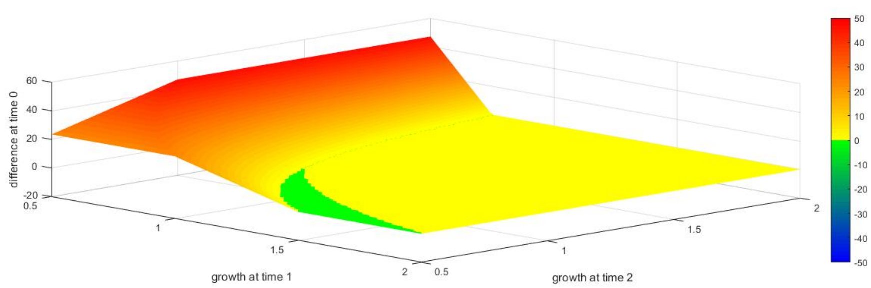
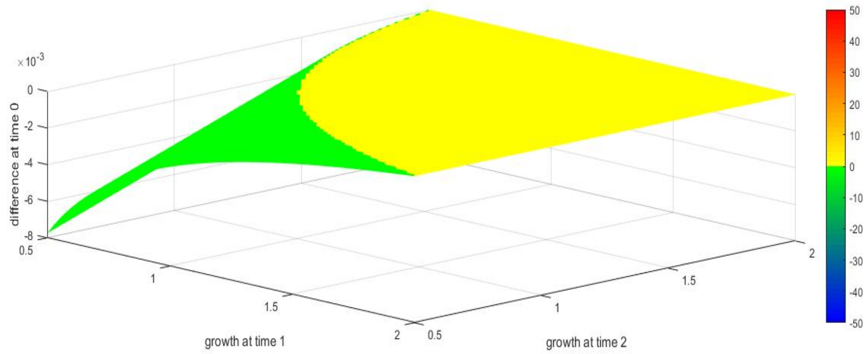


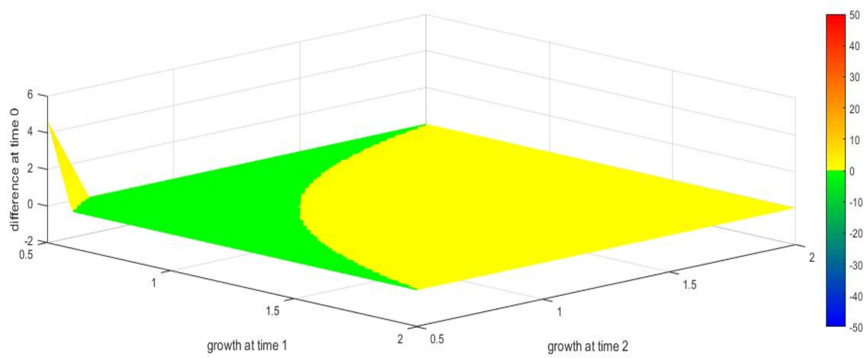
Figure 1: The x-axis and the y-axis both start from 0.5 and end at 2, because these are the 0.1% and 99.9% value of the log-normal distribution $e^{r-0.5\sigma^2+\sigma N}$, where N is a standard normal random variable. The z-axis is $V_w - V_w^*$, which is the difference between the value of both riders combined after making and not making the withdrawal at time 1. The color bar on the right indicates the corresponding value for different surface colors. Any colors from yellow to red indicates non-negative values, and colors from green to blue indicates negative values.

We first notice that the surface is generally above 0, with a small area of values smaller but close to 0. This means that the withdrawal decision at time 1 is very unlikely to decrease the total value of both riders under our specification. Only in some extreme cases, where R_1 is large and R_2 is small will $V_w - V_w^*$ be negative.

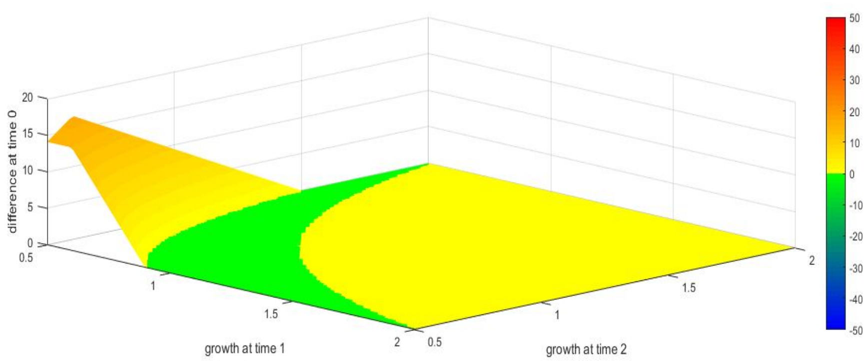
We are curious about how the withdrawal rate might change the look of the surface. Thus we continue to draw the same graph with X_w taking values from 10% to 90%, and the results are in Figure 2



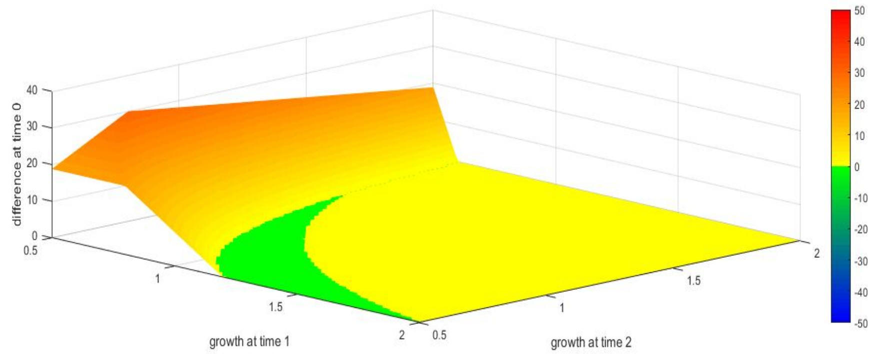
(a) $X_w = 10\%$



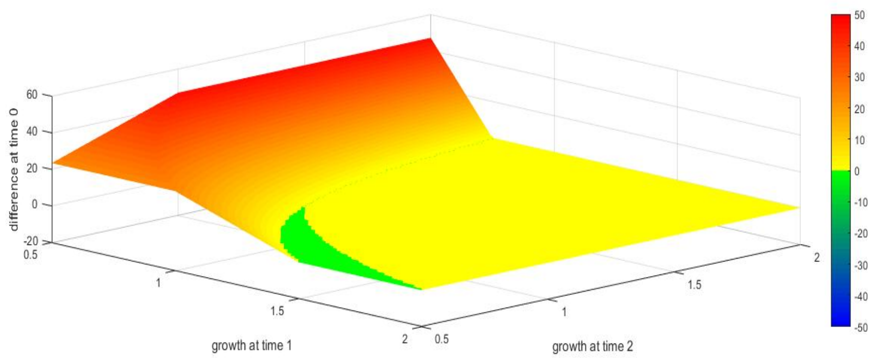
(b) $X_w = 20\%$



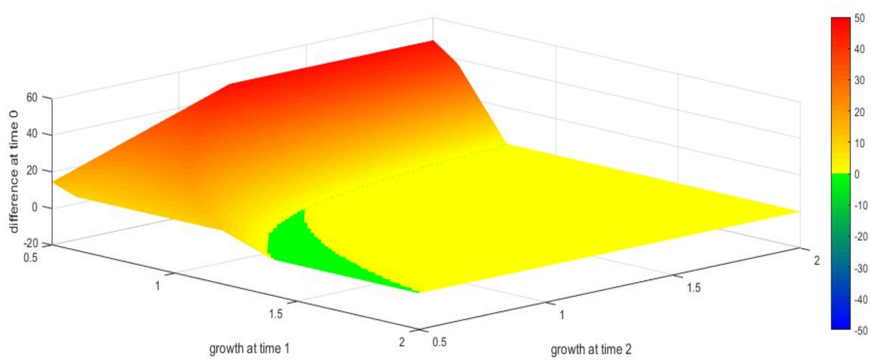
(c) $X_w = 30\%$



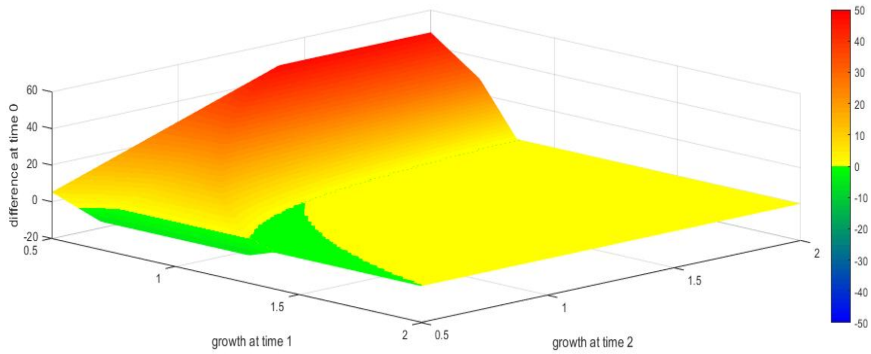
(d) $X_w = 40\%$



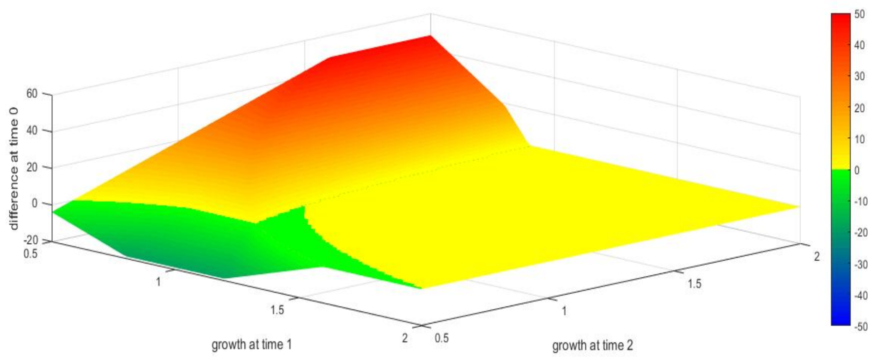
(e) $X_w = 50\%$



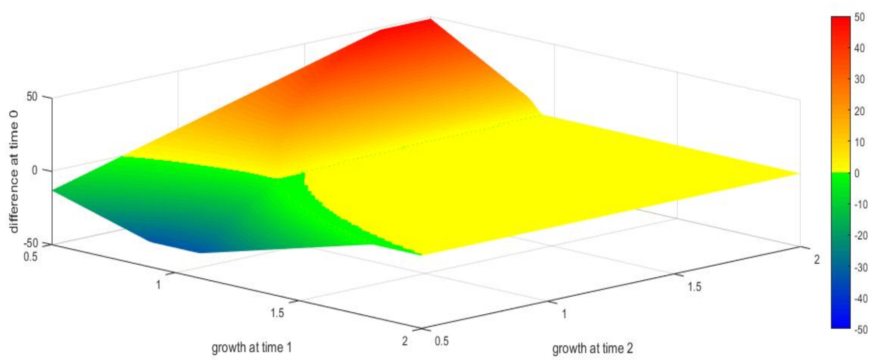
(f) $X_w = 60\%$



(g) $X_w = 70\%$



(h) $X_w = 80\%$



(i) $X_w = 90\%$

Figure 2: Xw changes the shape of the surface and the range of values

As can be seen, the shape of the surface looks very different when X_w is less than 20%. We observe that there are large areas of negative values, meaning that the withdrawal at time 1 decreased the total value of both riders. This is possibly due to the fact that when the withdrawal rate is small, it is unlikely that the GMWB will generate value, while the GMDB will. Since withdrawal decisions and GMDB values are negatively correlated, the withdrawal at time 1 has no GMWB value and decreased GMDB value at time 2.

When X_w is above 40%, the shape of the surface remains generally unchanged. We notice the negative area first shrinks and then grows, and at 90% withdrawal rate we would be able to reach very negative results. In fact, along with the growth of withdrawal rate, the graph has more extreme values appearing on both ends. This seems counter-intuitive at the beginning, since GMDB value should not create such difference. So why is it that the withdrawal at time 1 has reduced a great deal of the rider value?

Imagine an extreme case where the withdrawal rate is very close to 100%, meaning that the policyholder practically surrenders the policy should he chooses to withdrawal at any time. When he thus chooses, he will be entitled the greater of the investment-linked account value or the GMWB guaranteed value. By our definition of value, we know that unless the investment-link fund fails to achieve any kind of growth over the period, the GMWB will have 0 value since it only offers no more than A_0. This means that, withdrawing at time one is denying the potential growth in the investment-link account value or GMDB value at time 2. On the contrary, not withdrawing at time 1 and withdrawing at time 2 will allow the policyholder to get protection against further decline at time 2, thereby potentially generating huge GMWB value while not giving up GMDB value. This reasoning can be backed up by observing that the lowest values on the surface seems to appear in areas where both R_1 and R_2 are small, especially R_2.

For the moment, we have seen that under different withdrawal rates, there is a potential that not withdrawing at certain times will enable greater total values of the riders. We need to extend this model to a more commonly seen ten-period contract to verify our conclusions.

What is also worth noting is that, the surface area is not a direct translation of the actual probability of the underlying events. As the x-axis and y-axis are the exponential values taken from the log-normal distribution and arranged numerically, areas in the middle generally have a higher underlying probability density function compared to those in the corners.

3.2 The ten-period model

As mentioned in Chapter 2.2, we are assigning a dummy variable K_t to each withdrawal decision with $K_t = 1$ representing that the policyholder makes the withdrawal. This requires that we exhaust all possible withdrawal strategies to find out the optimal one. We are using Monte Carlo simulation to achieve this. Given the maturity T, we have 2^T different strategies for a single contract. The following are the specifications for the simulated contract

number of sample paths	gender	age	premium	maturity
10,000	female	40	20,000	10

Table 1: premium is set to be 20,000 so that the results does not bother too much with smaller digits. The sample paths for the investment-link fund for different withdrawal strategies are not repetitive, meaning that strategy 1 and strategy 2 have independent sample paths. We set seed (1) in the beginning to insure reproducibility

We first use a withdrawal rate $X_w = 1/T = 10\%$. Risk free rate and stock volatility are as before $r = 0.03$, $\sigma = 0.2$. The results are shown in Figure 3 and Table 2

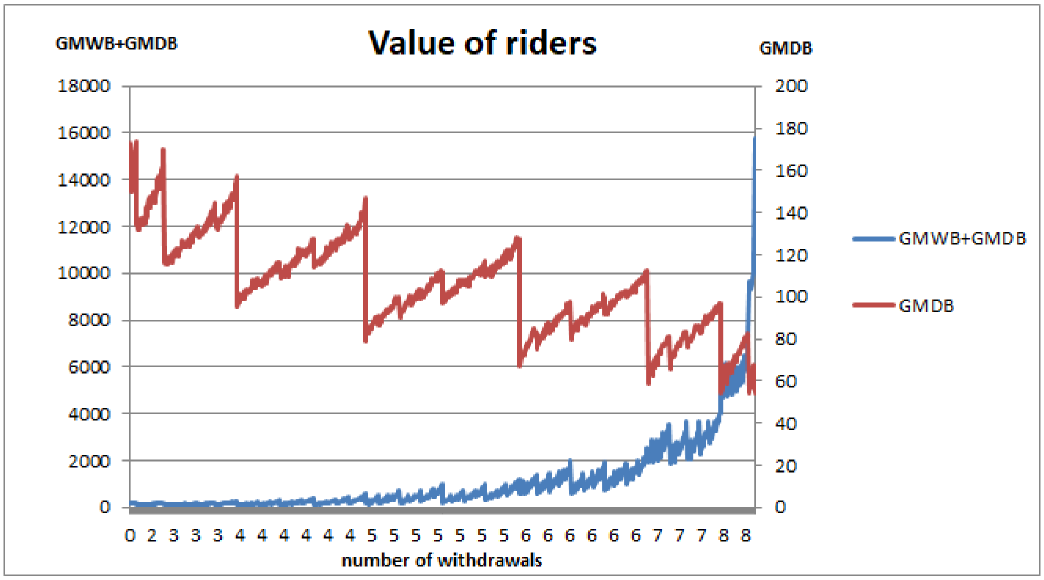


Figure 3: The x axis is the number of withdrawals actually made in each strategy, and altogether there are $2^{10} = 1024$ different strategies. The order within the same number of withdrawals is those that withdraws the earliest are on the left, e.g. for strategies with only one withdrawal but at different time, the one that withdraws at time 1 is the left most, and the one that withdraws at time 10 is the right most, followed by the strategy with 2 withdrawals at time 1 and time 2. The y axis on the left is the value for GMDB, and the right is GMWB+GMDB

number of withdrawals	max(std)	min(std)
0	173(2)	173(2)
1	174(2)	150(2)
2	184(4)	132(2)
3	270(13)	115(1)
4	552(26)	97(1)
5	1184(48)	147(9)
6	2501(77)	551(26)
7	4179(106)	1834(59)
8	7262(148)	4687(108)
9	10929(185)	9338(161)
10	**15772(223)**	15772(223)

Table 2: range of values for different withdrawal strategies with $X_w = 10\%$. Bold font is the global maximum value

The value of GMWB is much greater than the value of GMDB. Strategies with more than 5 withdrawals have a total value ranging from several thousand to sixteen thousand, while the GMDB value is always less than 200. The maximum value is obtained by withdrawing 10 out of 10 times. And the more we withdraw, the higher the GMWB value is. These results are consistent with what we have observed in the two-period model.

An unexplained pattern in Figure 3 is the irregular curve of GMDB value. It seems that the way we arrange the withdrawal strategy has an direct influence on why this pattern exists. First, we notice that GMDB value is increasing within the group of strategies with the same number of withdrawals and decreasing with the number of withdrawals. To understand this, we need to know the order of the withdrawal strategy. We use strategies with 9 withdrawals as an example to demonstrate the way the strategies are ordered (Table 3), with 1 representing withdrawing at time t, and 0 representing not withdrawing at time t. This means the strategies that withdraws the earliest is on the left in Figure 3. Therefore, this tells us that by postponing the withdrawals, the value of both GMWB+GMDB is increased.

Total Value(GMDB)	withdrawal decision at t									
	1	2	3	4	5	6	7	8	9	10
9412(54)	1	1	1	1	1	1	1	1	1	0
9659(56)	1	1	1	1	1	1	1	1	0	1
9452(57)	1	1	1	1	1	1	1	0	1	1
9402(58)	1	1	1	1	1	1	0	1	1	1
9338(61)	1	1	1	1	1	0	1	1	1	1
9594(64)	1	1	1	1	0	1	1	1	1	1
9704(65)	1	1	1	0	1	1	1	1	1	1
10034(66)	1	1	0	1	1	1	1	1	1	1
10234(67)	1	0	1	1	1	1	1	1	1	1
10929(68)	0	1	1	1	1	1	1	1	1	1

Table 3: withdrawal strategy order

The reasoning behind this is as follows. For the GMWB, given that the investment account value diverges as we go further into the life of the contract, it is more likely that the investment account will reach extreme values in later stages. Since GMWB is essentially an insurance against the loss of investment account values over time, the longer we postpone the withdrawals, the higher the withdrawal benefit could be. The same reasoning can be applied to GMDB. Recalling that the guaranteed death benefit account will be adjusted pro rata after withdrawals are made, withdrawing early will inevitably decrease the GMDB value in every period afterwards.

So far we seem to have concluded that the withdrawing all the time is still the optimal strategy. But is it true for all X_w's? We know from the two period model that withdrawal rate could negatively influence the total value of both riders. Therefore we set the withdrawal rate to be values other than $1/T$ to see if the conclusion about the optimal strategy holds. The following are the results.

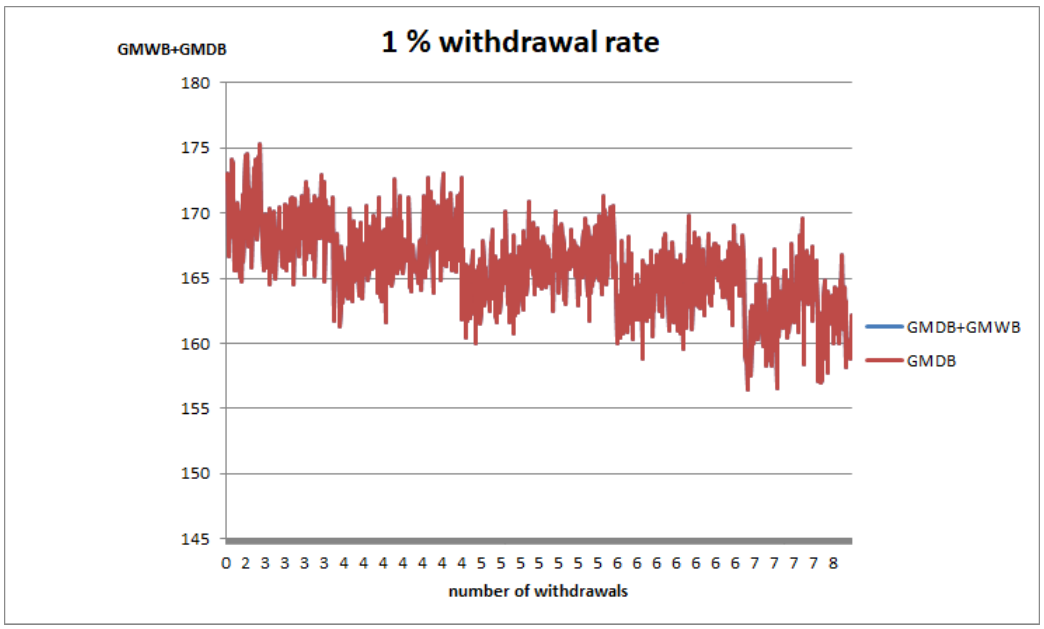


Figure 4: $X_w = 1\%$

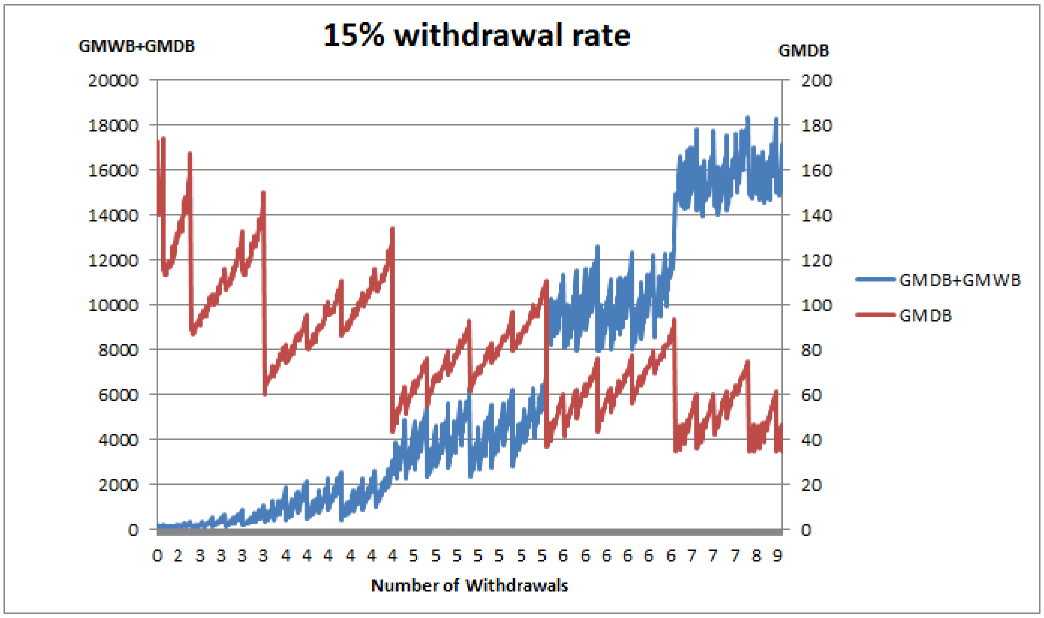


Figure 5: $X_w = 15\%$

number of withdrawals	max(std)	min(std)
0	173 (2)	173 (2)
1	177 (2)	140 (3)
2	297 (1)	113 (14)
3	1063 (1)	91 (44)
4	3073 (20)	319 (89)
5	6790 (68)	2267 (149)
6	13320 (141)	7909 (223)
7	**18365 (200)**	13952 (263)
8	18276 (204)	14525 (259)
9	17118 (209)	14852 (241)
10	15336(218)	15336 (218)

Table 4: range of values for different withdrawal strategies with $X_w = 15\%$. Bold font is the global maximum value

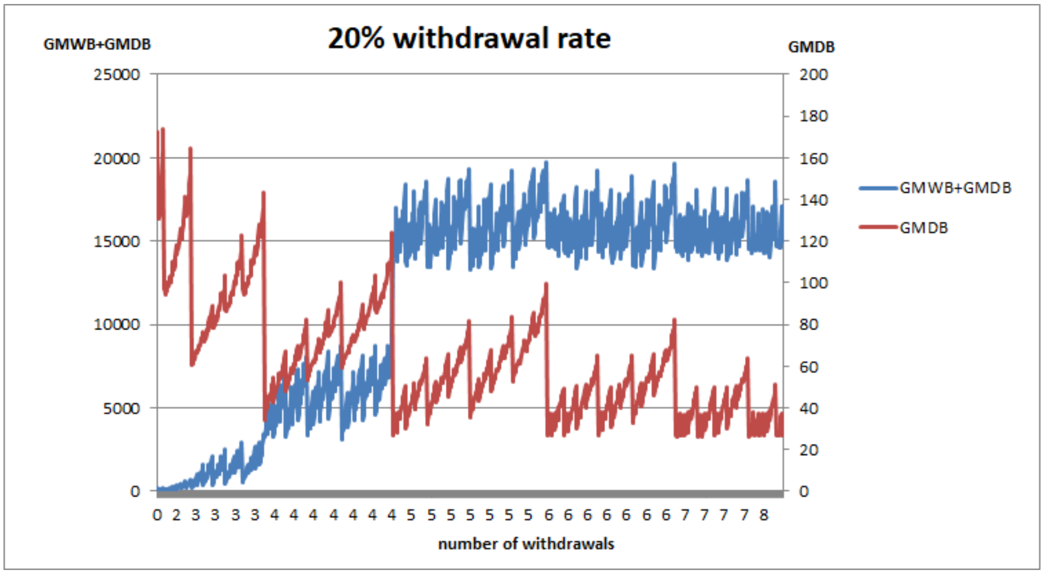


Figure 6: $X_w = 20\%$

number of withdrawals	max(std)	min(std)
0	173(2)	173(2)
1	189(2)	131(4)
2	733(1)	98(33)
3	3413(14)	215(98)
4	9738(76)	3126(186)
5	**19743(188)**	13250(285)
6	19637(189)	13389(284)
7	18656(196)	13846(268)
8	18604(199)	14058(263)
9	17086(205)	14579(241)
10	14917(212)	14917(212)

Table 5: range of values for different withdrawal strategies with $X_w = 20\%$. Bold font is the global maximum value

GMWB+GMDB	withdrawal decision at t									
	1	2	3	4	5	6	7	8	9	10
19743	0	0	0	0	0	1	1	1	1	1

Table 6: the maximum GMWB+GMDB value and its corresponding withdrawal strategy

When X_w is very small, the GMWB would have very little or no effect and thus GMDB is the only rider that is valuable. In our case where $X_w = 1\%$, the GMWB has 0 value the whole time: all 10 periods over 10,000 sample paths. Therefore we choose to never withdrawal in order to get full GMDB value.

What interests us is when $X_w > 10\%$. In this case, the optimal strategy is no longer the strategy that withdraws all the time. In fact, when $X_w = 15\%$, the value reaches its maximum at 7 withdrawals, and then slowly declines at 8, 9 and 10. The reasoning is as follows. Since $7 \times 0.15 = 1.05 > 1$, withdrawing equal or more than 7 times will provide the same result, that is the guaranteed withdrawal account will certainly be depleted, and therefore the amount that is withdrawn from these strategies are all capped at A_0. This means that any withdrawals made after the 7^{th} withdrawal will have no additional value (because they are zero), and only reduces the GMDB values. This also means that withdrawing from time 3 to time 9 for a total of 7 withdrawals generates approximately the same GMWB values as withdrawing from time 3 to time 10 for a total of 8 withdrawals, even though the latter has one more withdrawal attempt in the last period. More importantly, as we have discussed, with a fixed number of withdrawal opportunities, the policyholder is tempted to postpone the withdraws as much as possible, in order to get as much protection as possible against the underside risk of the investment account. The optimal strategy is thus starting withdrawing at time 4 until maturity, with a total withdrawal of 7 times.

We can see that when $X_w = 20\%$ (Figure 6 and Table 5) the optimal strategy is now to withdraw 5 times. The order of withdrawal is shown in Table 6. This confirms our previous analysis.

4 Conclusion

In this paper, we explored the interactive relationship of the value between GMWB and GMDB in a semi-static model and decided whether it is always optimal to withdraw all the time under different withdrawal rates.

Our conclusion is, when the withdrawal rate is $1/T$, it is optimal to withdraw all the time and the value of GMWB is much greater than that of GMDB in the optimal strategy. However, when the withdrawal rate is much lower or higher than $1/T$, withdrawing all the time is no longer the best strategy. Specifically, when withdrawal rate is very low, the value of GMDB is greater than that of GMWB and therefore not withdrawing at all is the best strategy. If withdrawal rate is higher than $1/T$, then the smallest integer n that gives $nX_w \geq 1$ is the number of withdrawals to be made in the optimal strategy. In addition, the withdrawals starts at $T - n$.

From what we have discovered in this paper, we speculate that with different configurations of other factors, such as mortality rate and risk-free interest rate, we might also come up with optimal strategies that are not making constant withdrawals. This is worth studying from the risk management perspective, since our usual assumptions about the different parameters of the contract and policyholder behaviors are not exactly realistic all the time.

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