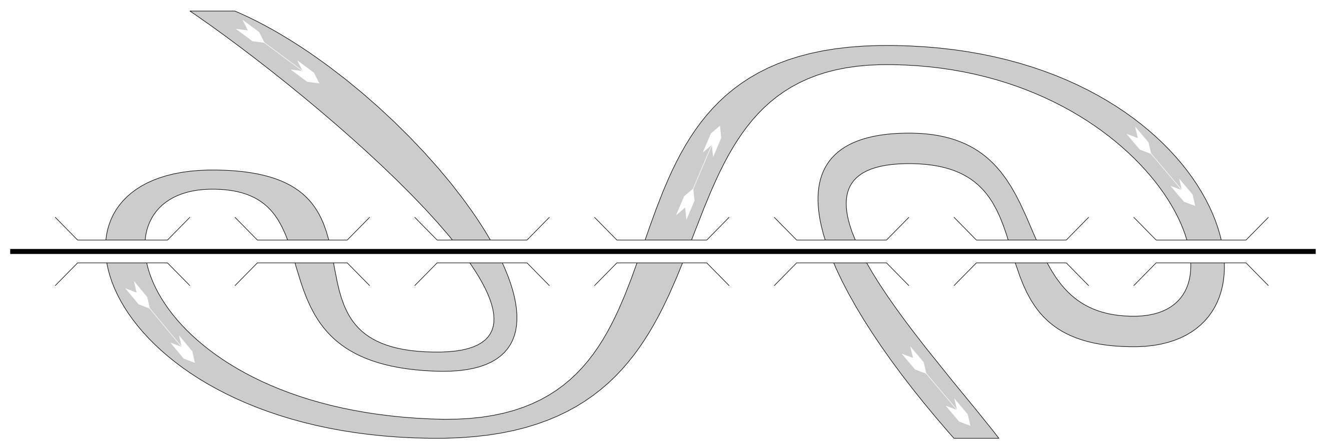
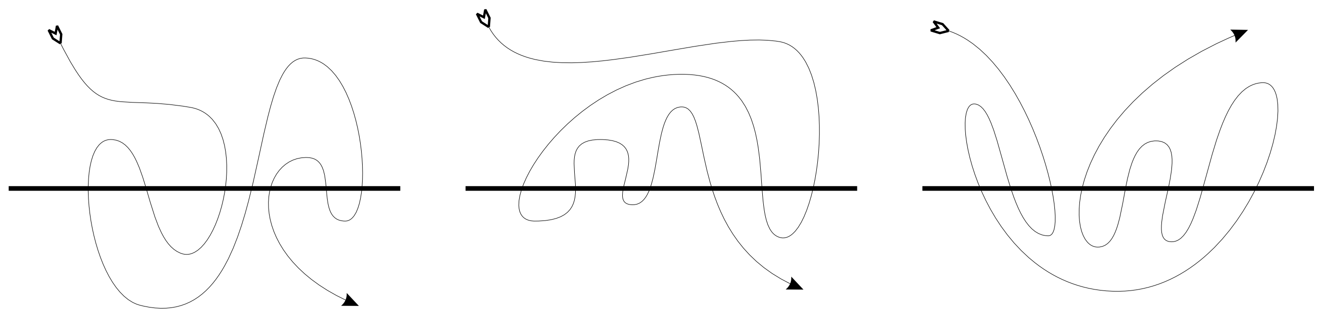
Approaches to the Enumerative Theory of Meanders

1. Definitions

meanders 1. Sinuous windings (of a river). 2. Ornamental pattern of lines winding in and out. [From the Greek $\mu\alpha\iota\alpha\nu\delta\rho o\sigma$, appellative use of the name of a river in Phrygia noted for its winding course.]



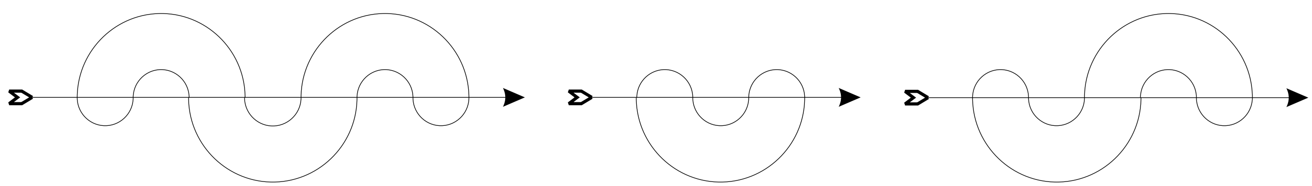
Definition 1. *An* open meander *is a configuration consisting of an oriented simple curve and a line in the plane, that cross a finite number of times and intersect only transversally. Two open meanders are equivalent if there is a homeomorphism of the plane that maps one meander to the other.*



$n = 7$ $n = 7$ $n = 8$

The number of crossings between the two curves is the *order* of a meander.

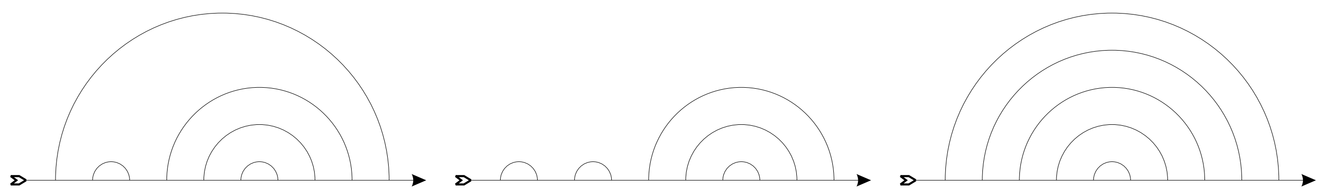
Definition 2. *A* (closed) meander *is a planar configuration consisting of a simple closed curve and an oriented line, that cross finitely many times and intersect only transversally. Two meanders are equivalent if there exists a homeomorphism of the plane that maps one to the other.*



$n = 4$ $n = 2$ $n = 3$

The *order* of a closed meander is defined as the number of pairs of intersections between the closed curve and the line.

Definition 3. *An* arch configuration *is a planar configuration consisting of pairwise non-intersecting semicircular arches lying on the same side of an oriented line, arranged such that the feet of the arches are equally spaced along the line.*



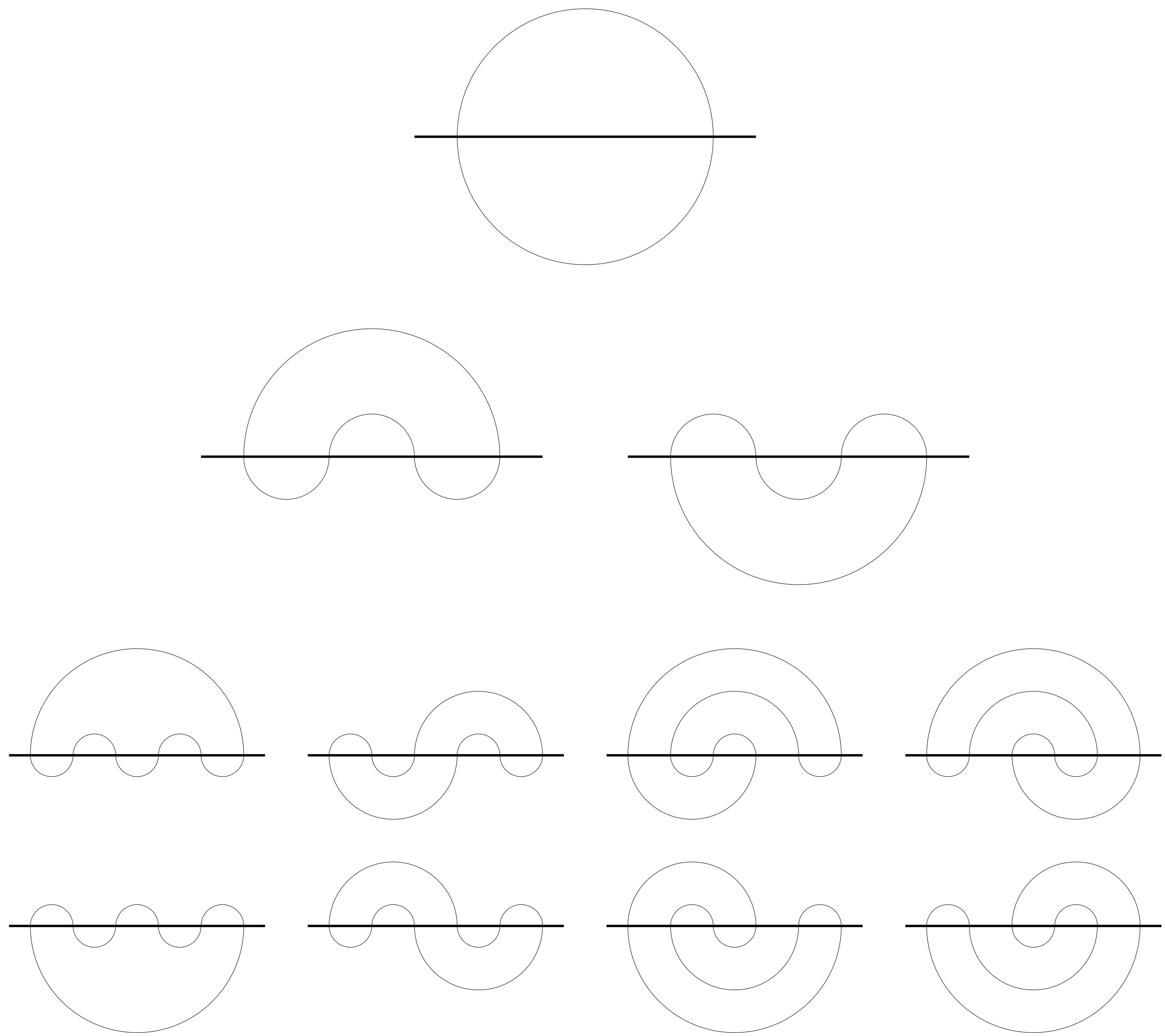
The *order* of an arch configuration is the number of arch configurations it contains. There are $C_n = \frac{1}{n+1}\binom{2n}{n}$

arch configurations of order n.

The Enumerative Problem

The enumerative problem associated with meanders is to determine

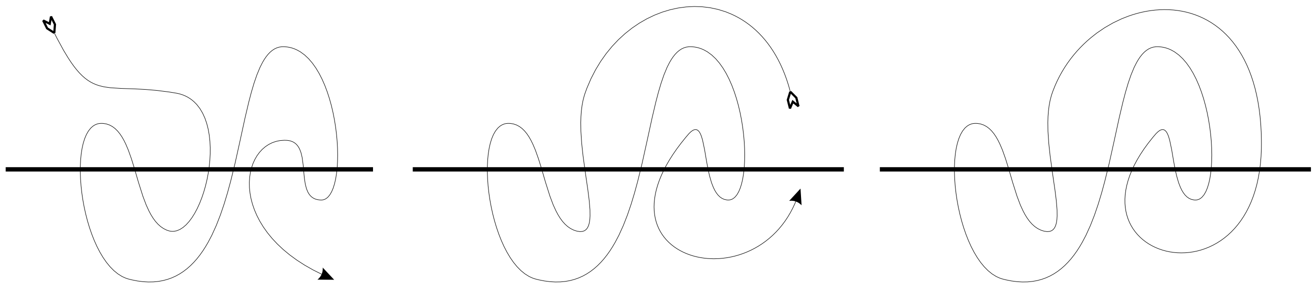
- m_n, the number of inequivalent open meanders of order n, and
- M_n, the number of inequivalent closed meanders of order n.



n	M_n	n	M_n	n	M_n
1	1	9	933458	17	59923200729046
2	2	10	8152860	18	608188709574124
3	8	11	73424650	19	6234277838531806
4	42	12	678390116	20	64477712119584604
5	262	13	6405031050	21	672265814872772972
6	1828	14	61606881612	22	7060941974458061392
7	13820	15	602188541928	23	74661728661167809752
8	110954	16	5969806669034	24	794337831754564188184

Table 1: The first 24 meandric numbers.

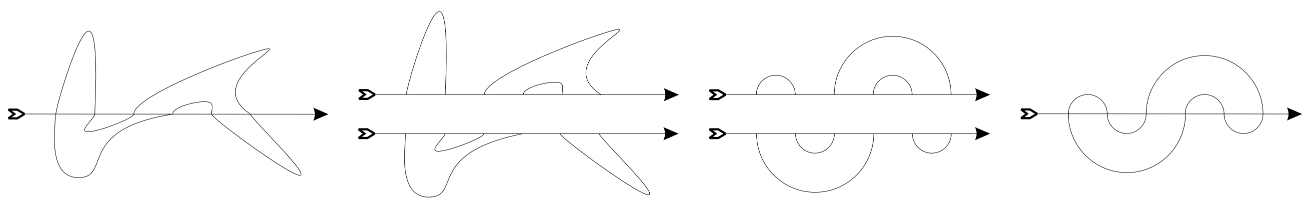
The two problems are related by $M_n = m_{2n-1}$.



The rest of the talk will focus on constructions dealing with M_n.

A Canonical Form

Define a unique representative for each class of meanders.



The line cuts a meander into two components, each of which can be encoded as an arch configuration.

A meandric system is the superposition of an arbitrary upper and lower arch configuration of the same order. There are

$$\left(\frac{1}{n+1}\binom{2n}{n}\right)^2$$

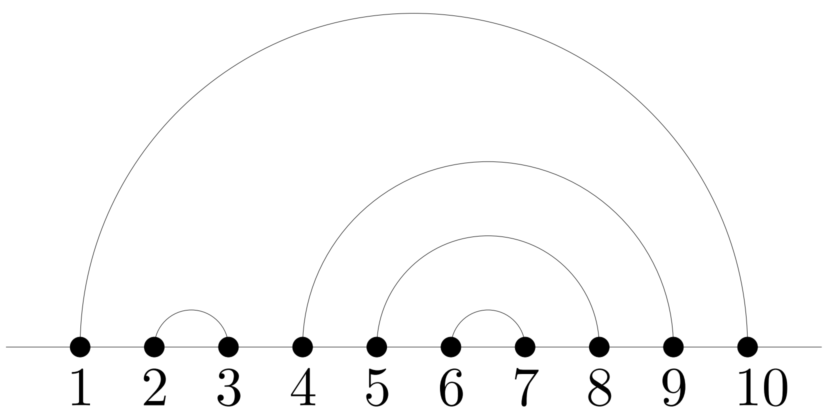
meandric systems of order n. Use $M_n^{(k)}$ to denote the number of k

component meandric systems of order n. This is a natural generalization of meandric numbers, in the sense that $M_n = M_n^{(1)}$.

2. Using Group Characters

Arch Configurations as Elements of $\mathfrak{S}_{2n}$

Each arch is encoded as a transposition on its basepoints. An arch configuration encoded as the product these transpositions.



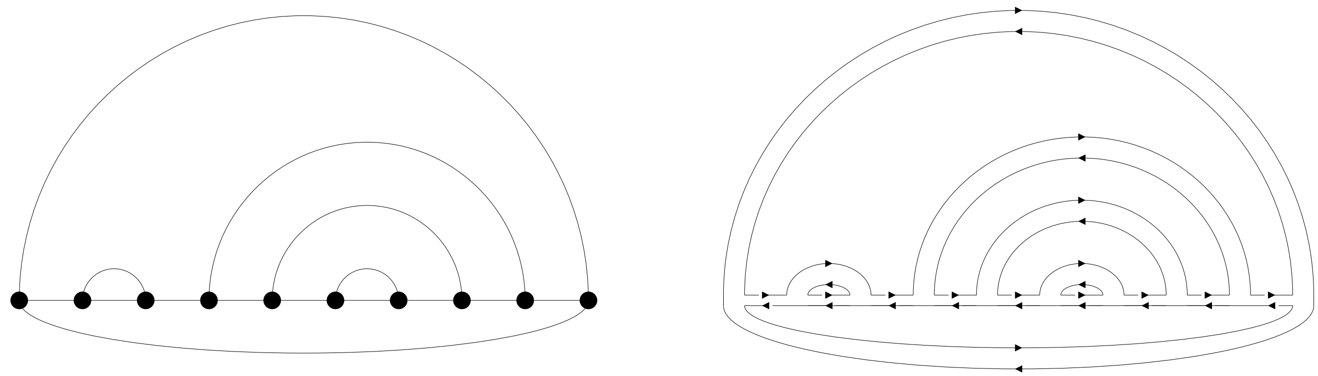
The arch configuration $(1\ 10)(2\ 3)(4\ 9)(5\ 8)(6\ 7)$

Lemma 2.1. *The permutations in* $\mathfrak{S}_{2n}$ *that correspond to arch configurations form the set*

$$\{\mu \in \mathcal{C}_{(2^n)} \colon \kappa(\sigma\mu) = n+1\},$$

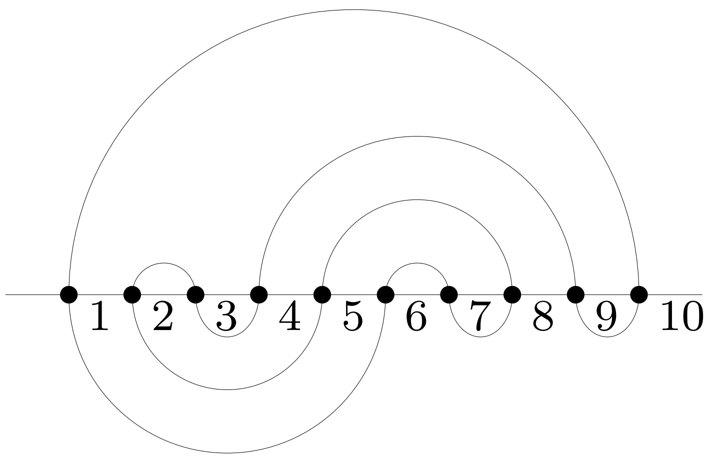
where $\kappa(\pi)$ *denote the number of cycles in the disjoint cycle representation of the permutations* π*, and* $\sigma = \sigma_n = (1\ 2\ \ldots\ 2n)$*.*

Proof. Encode the arch configurations as a graph.



The faces of the map are the cycles of $\sigma\mu$ plus an extra face defined by the lower half plane. Since the graph is connected with $2n$ edges and $3n$ vertices, it is planar if and only if $\kappa(\sigma\mu) = n + 1$. □

Lemma 2.2. *If (μ_1, μ_2) is an ordered pair of transposition representations of arch configurations of order n, then the meandric system for which μ_1 represents the upper configuration and μ_2 represents the lower configuration is a meander if and only if $\mu_1\mu_2 \in \mathcal{C}_{(n^2)}$.*



Proof. The cycles of $\mu_1\mu_2$ are obtained by following the curve two steps at a time. Thus $\mu_1\mu_2$ is the square of a permutation in the conjugacy class $\mathcal{C}_{(2n)}$. □

Corollary 2.3. *The class of meanders of order n is in bijective correspondence with the set*

$$\left\{ \begin{aligned} (\mu_1, \mu_2) \in \mathcal{C}_{(2^n)} \times \mathcal{C}_{(2^n)} \colon \ \kappa(\sigma\mu_1) = \kappa(\sigma\mu_2) = n+1, \\ \mu_1\mu_2 \in \mathcal{C}_{(n^2)}. \end{aligned} \right\}$$

This gives us the expression

$$M_n = \sum_{\lambda_1, \lambda_2 \in \mathcal{C}^{(n+1)}} \sum_{\mu_1, \mu_2 \in \mathcal{C}_{(2^n)}} \delta_{[\sigma\mu_1], \lambda_1} \delta_{[\sigma\mu_2], \lambda_2} \delta_{[\mu_1\mu_2], (n^2)} \qquad (1)$$

Using the orthogonality relation

$$\delta_{[\lambda],[\mu]} = \frac{|[\lambda]|}{(2n)!} \sum_{i=1}^{k} \chi^{(i)}(\lambda)\chi^{(i)}(\mu), \tag{2}$$

where $\chi^{(i)}$, $\chi^{(2)}$, ..., $\chi^{(k)}$ are the characters of the irreducible representations of $\mathfrak{S}_{2n}$, gives

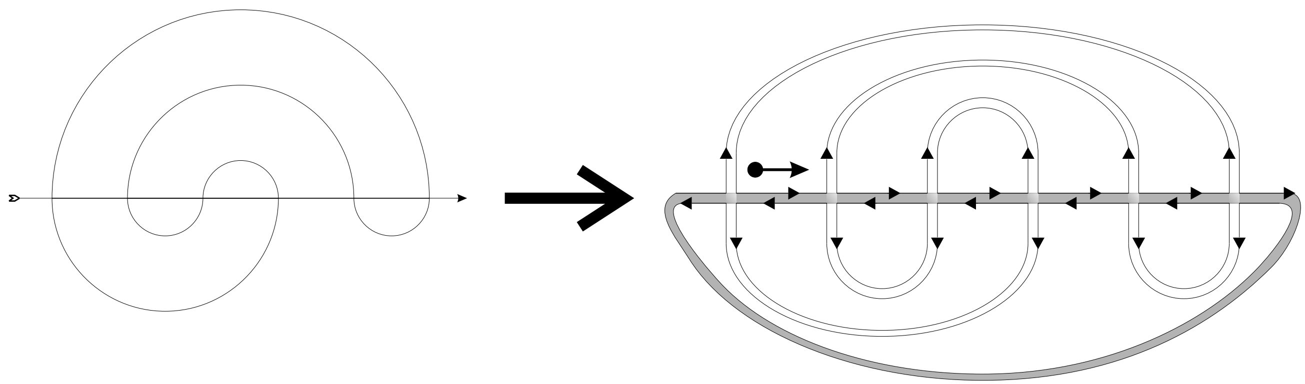
$$M_n = \sum_{\lambda_1,\lambda_2 \in \mathcal{C}^{(n+1)}} \frac{|(2^n)|^2 \cdot |\lambda_1| \cdot |\lambda_2| \cdot |(n^2)|}{((2n)!)^5} \sum_{\mu_1,\mu_2 \in \mathfrak{S}_{2n}} \sum_{f,g,h,i,j=1}^{k} \chi^{(f)}(\mu_1)\chi^{(f)}(2^n)\chi^{(g)}(\mu_2)\chi^{(g)}(2^n)\chi^{(h)}(\sigma\mu_1)\chi^{(h)}(\lambda_1) \times \chi^{(i)}(\sigma\mu_2)\chi^{(i)}(\lambda_2)\chi^{(j)}(\mu_1\mu_2)\chi^{(j)}(n^2) \tag{3}$$

3. Using Matrix Integrals

The Encoding

Meanders are encoded as 4-regular maps. The resulting maps are a subclass of $\mathcal{R}$ defined as follows.

Definition 4. *The class* $\mathcal{R}$ *is the class of oriented* 4*-regular ribbon graphs on labelled vertices, with edges divided into two classes, such that around every vertex the edges alternate between the two classes. For each vertex, one of the edges of the second class is designated as up.*



Lemma 3.1. *Meanders of order n are in $4n$ to $(2n)! \cdot 2^{2n}$ correspondence with graphs in $\mathcal{R}$ on $2n$ vertices that are genus zero and have exactly two cycles induced by the partitioning of the edges, one of each class.*

The Goal

Using $\mathcal{R}_m$ to denote the elements of $\mathcal{R}$ with m vertices, and $p(G)$, and $r(G)$ to denote, respectively, the number of faces of G, and the number of cycles induced by the edge colouring, in G, Lemma 3.1, lets us recover M_n from the generating series

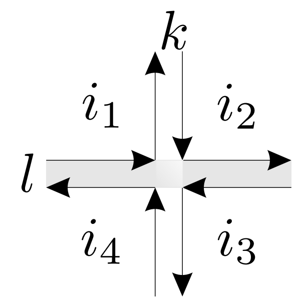
$$Z(s,q,N) = \frac{1}{N^2} \sum_{m\geq 1} \frac{(-1)^m}{m!} \frac{s^m}{N^m} \sum_{G\in\mathcal{R}_m} N^{p(G)} q^{r(G)}. \tag{4}$$

Using the expression

$$\frac{2^{2n}}{4n} M_n = [s^{2n}] \lim_{N\to\infty} [q^2] Z(s,q,N). \tag{5}$$

Determining $Z(s,q,N)$

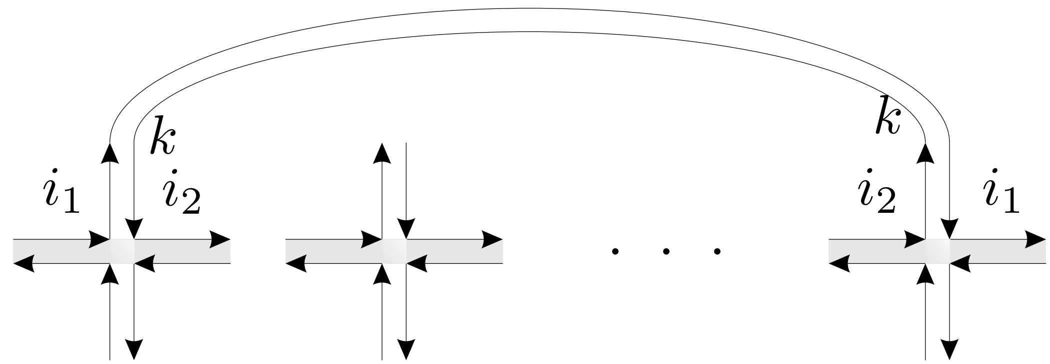
- $N^{p(G)}q^{r(G)}$ is the number of ways to colour the faces of G from $\{1,2,\ldots,N\}$ and the induces cycles from $\{1,2,\ldots,q\}$.
- Consider vertex neighbourhoods of coloured elements of $\mathcal{R}$.
- Use triply indexed variables h and g to label such a neighbourhood by $h^{(k)}_{i_1 i_2} g^{(l)}_{i_2 i_3} h^{(k)}_{i_3 i_4} g^{(l)}_{i_4 i_1}$. Use $\mathbf{H}_k$ for the matrix $\left(h^{(k)}_{ij}\right)$ and $\mathbf{G}_k$ for $\left(g^{(k)}_{ij}\right)$.



- The sum over all possible neighbourhoods is

$$\sum_{k,l=1}^{q} \sum_{i_1,i_2,i_3,i_4=1}^{N} h^{(k)}_{i_1 i_2} g^{(l)}_{i_2 i_3} h^{(k)}_{i_3 i_4} g^{(l)}_{i_4 i_1} = \operatorname{tr} \sum_{k,l=1}^{q} \mathbf{H}_k \mathbf{G}_l \mathbf{H}_k \mathbf{G}_l$$

Given m vertex neighbourhoods, construct a map by matching the half-edges.



Every variable $h_{ij}^{(k)}$ must be paired with $h_{ji}^{(k)}$ and every variable $g_{ij}^{(k)}$ must be paired with $g_{ji}^{(k)}$.

Maps in $\mathcal{R}_m$ correspond to matchings of the variables in the expression

$$\left(\operatorname{tr}\sum_{k,l=1}^{q}\mathbf{H}_k\mathbf{G}_l\mathbf{H}_k\mathbf{G}_l\right)^m.$$

Gaussian Measures

For $\mathbf{B}$ a positive definite matrix giving a quadratic form on $\mathbb{R}^n$,

$$\mathrm{d}\mu(x) = (2\pi)^{-n/2}(\det \mathbf{B})^{1/2} \exp\left(-\tfrac{1}{2}x^T\mathbf{B}x\right) \mathrm{d}x,$$

is the *Gaussian measure* on $\mathbb{R}^n$ associated with $\mathbf{B}$, where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^n$.

For a function f,

$$\langle f \rangle = \int_{\mathbb{R}^n} f(x)\mathrm{d}\mu(x)$$

is the average value of f with respect to $\mathrm{d}\mu$.

Determining $\langle f \rangle$

Lemma 3.2. *If $x_1, x_2, \ldots, x_n$ are the coordinates on $\mathbb{R}^n$, then*

$$\langle x_i x_j \rangle = a_{ij},$$

for all $1 \leq i, j \leq n$, where $\mathbf{A} = (a_{ij}) = \mathbf{B}^{-1}$.

Lemma 3.3 (Wick's Formula). *If $f_1, f_2, \ldots, f_{2m}$ are linear functions of x, then*

$$\langle f_1 f_2 \cdots f_{2m} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \cdots \langle f_{p_m} f_{q_m} \rangle,$$

with the sum over all matchings of $\{1, 2, \ldots, 2m\}$ into pairs $\{p_i, q_i\}$.

The space $\mathcal{H}_N$ of $N \times N$ Hermitian matrices is isomorphic to $\mathbb{R}^{N^2}$. $\mathrm{tr}(\mathbf{H}^2)$ is a positive definite quadratic form on $\mathcal{H}_N$. The measure

$$d\mu(\mathbf{H}) = (2\pi)^{-N^2/2} 2^{(N^2-N)/2} \exp\left(-\tfrac{1}{2}\operatorname{tr}\mathbf{H}^2\right) dv(\mathbf{H}),$$

where

$$dv(\mathbf{H}) = \prod_{i=1}^{N} dx_{ii} \prod_{1 \le i < j \le N} dx_{ij} dy_{ij}$$

is the Lebesgue measure on the space gives

Lemma 3.4. $\operatorname{tr}\mathbf{H}^2$,

$$\langle h_{ij} h_{kl} \rangle = \delta_{i,l} \delta_{j,k}.$$

Extending this to a measure on the space $(\mathcal{H}_N)^{2q}$ of $2q$-tuples $(\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_q, \mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_q)$ of $N \times N$ Hermitian matrices gives

$$\left\langle \left(\operatorname{tr} \sum_{k,l=1}^{q} \mathbf{H}_k \mathbf{G}_l \mathbf{H}_k \mathbf{G}_l \right)^m \right\rangle = \sum_{G \in \mathcal{R}_m} N^{p(G)} q^{r(G)}. \tag{6}$$

$$
\begin{aligned}
Z(s,q,N) &= \frac{1}{N^2}\sum_{m\geq 0}\frac{(-1)^m}{m!}\frac{s^m}{N^m}\left\langle\left(\operatorname{tr}\sum_{k,l=1}^{q}\mathbf{H}_k\mathbf{G}_l\mathbf{H}_k\mathbf{G}_l\right)^m\right\rangle \\
&= \frac{1}{N^2}\left\langle\exp\left(-\frac{s}{N}\operatorname{tr}\sum_{k,l=1}^{q}\mathbf{H}_k\mathbf{G}_l\mathbf{H}_k\mathbf{G}_l\right)\right\rangle \\
&= \frac{1}{N^2}\int_{(\mathcal{H}_N)^{2q}}\exp\left(-\frac{s}{N}\operatorname{tr}\sum_{k,l=1}^{q}\mathbf{H}_k\mathbf{G}_l\mathbf{H}_k\mathbf{G}_l\right) \\
&\qquad \prod_{k,l=1}^{q}\mathrm{d}\mu(\mathbf{H}_k)\mathrm{d}\mu(\mathbf{G}_l).
\end{aligned}
\tag{7}
$$

4. The Temperley-Lieb Algebra

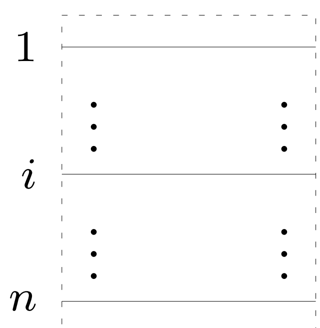
Definition 5. *The* Temperley-Lieb algebra *of order n in the indeterminate q, denoted $\mathfrak{TL}_n(q)$, is a free additive algebra over $\mathbb{C}(q)$ with multiplicative generators* 1, e_1, e_2, $\ldots$, e_{n-1} *subject to the relations:*

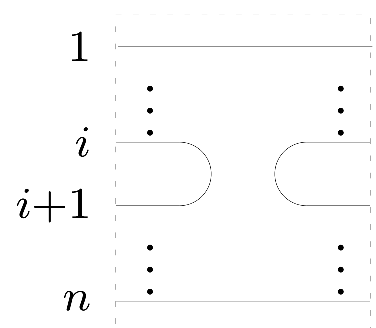
$$\mathsf{e}_i^2 = q\,\mathsf{e}_i \qquad \text{for } i = 1, 2, \ldots, n-1 \tag{R1}$$

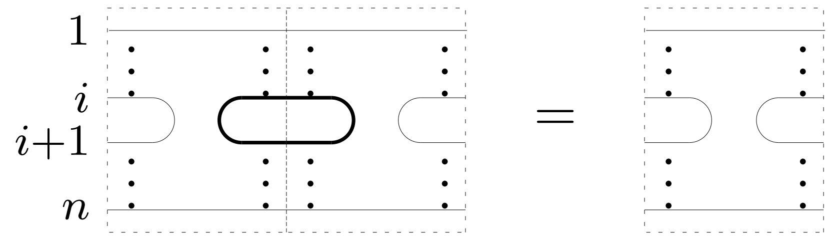
$$\mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i \qquad \text{if } |i-j| > 1 \tag{R2}$$

$$\mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i = \mathsf{e}_i \qquad \text{for } i = 1, 2, \ldots, n-1 \tag{R3}$$

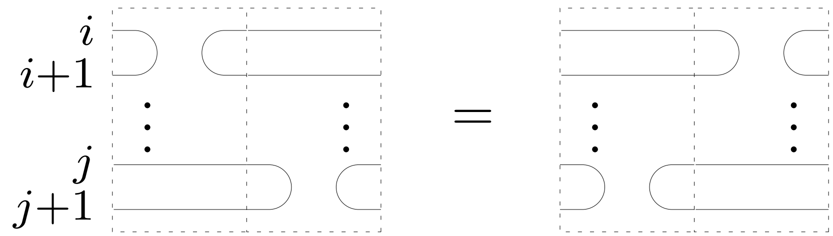
where 1 *is the multiplicative identity.*

$1 =$ 

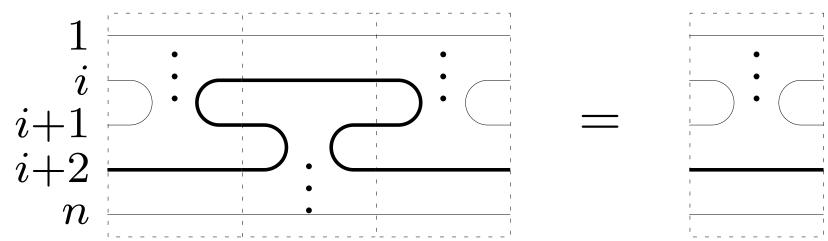
$\mathsf{e}_i =$ 

(R1) $\mathsf{e}_i =$ 

$= q\,\mathsf{e}_i$

(R2) $\mathsf{e}_i\mathsf{e}_j =$ 

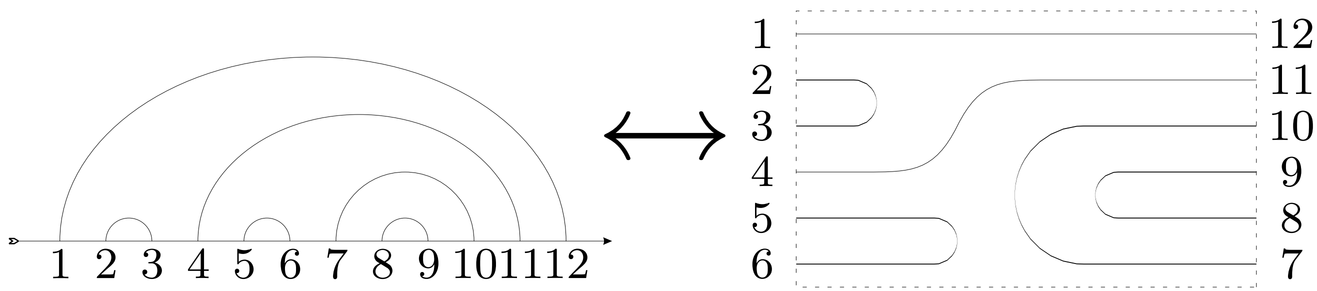
$= \mathsf{e}_j\mathsf{e}_i$

(R3) $\mathsf{e}_i\mathsf{e}_{i+1}\mathsf{e}_i =$ 

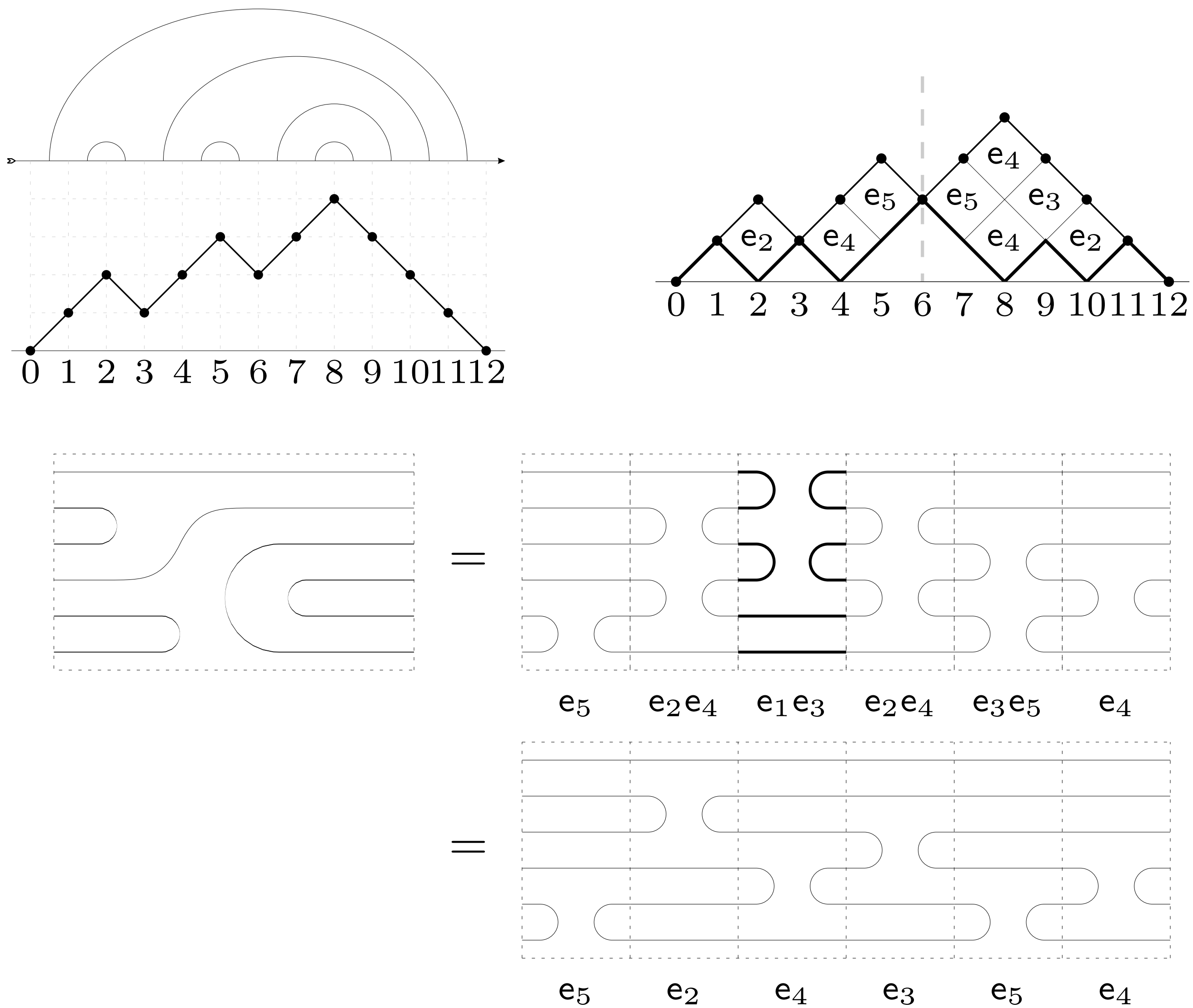
$= \mathsf{e}_i$

The Encoding

Arch configurations are encoded as strand diagrams.



These strand diagrams for a basis for the Temperley-Lieb algebra.



0 1 2 3 4 5 6 7 8 9 10 11 12
e2 e4 e5 e5 e4 e3 e4 e2
0 1 2 3 4 5 6 7 8 9 10 11 12
=
e5 e2e4 e1e3 e2e4 e3e5 e4
=
e5 e2 e4 e3 e5 e4

A Markov Trace

$\big(\mathfrak{TL}_1(q), \mathfrak{TL}_2(q), \dots\big)$ is a sequence of nested algebras, with $\mathfrak{TL}_n(q)$ embedded in $\mathfrak{TL}_{n+1}(q)$ by the map $\varphi\colon \mathfrak{TL}_n(q) \to \mathfrak{TL}_{n+1}(q), \mathsf{e}_i \mapsto \mathsf{e}_i$. extended as a homomorphism.

Definition 6. *A family of functions* $\mathrm{tr}_k\colon \mathfrak{TL}_k(q) \to \mathbb{C}(q)$ *for* $k \geq 1$ *is called a* Markov trace *if it satisfies:*

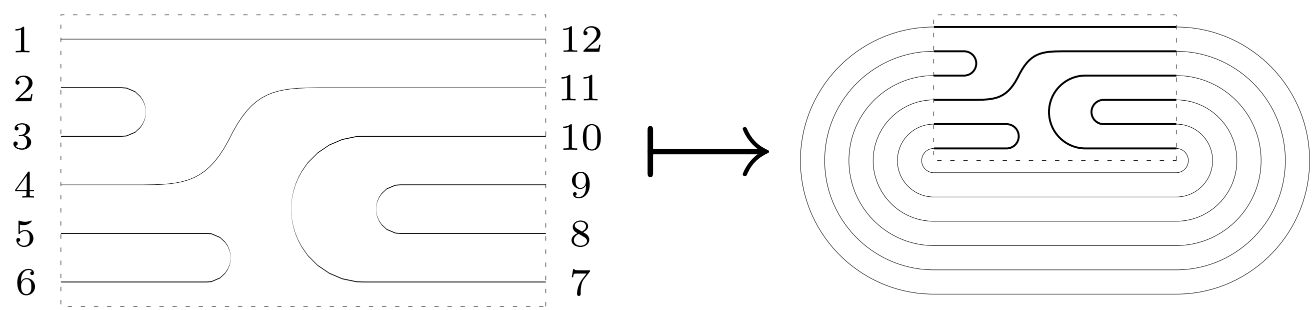
$$\mathrm{tr}_k \text{ is a linear functional,} \tag{T1}$$

$$\mathrm{tr}_k(\mathsf{ef}) = \mathrm{tr}_k(\mathsf{fe}) \text{ for all } \mathsf{e}, \mathsf{f} \in \mathfrak{TL}_k(q), \text{ and} \tag{T2}$$

$$\mathrm{tr}_{k+1}(\mathsf{e}\mathsf{e}_k) = \mathrm{tr}_k(\mathsf{e}) \text{ if } \mathsf{e} \in \langle 1, \mathsf{e}_1, \dots, \mathsf{e}_{k-1} \rangle. \tag{T3}$$

The Closure of a Strand Diagram

Close a strand diagram of order n by identifying the endpoint i with the endpoint $n{-}i{+}1$ for each i in the range $1 \leq i \leq n$.

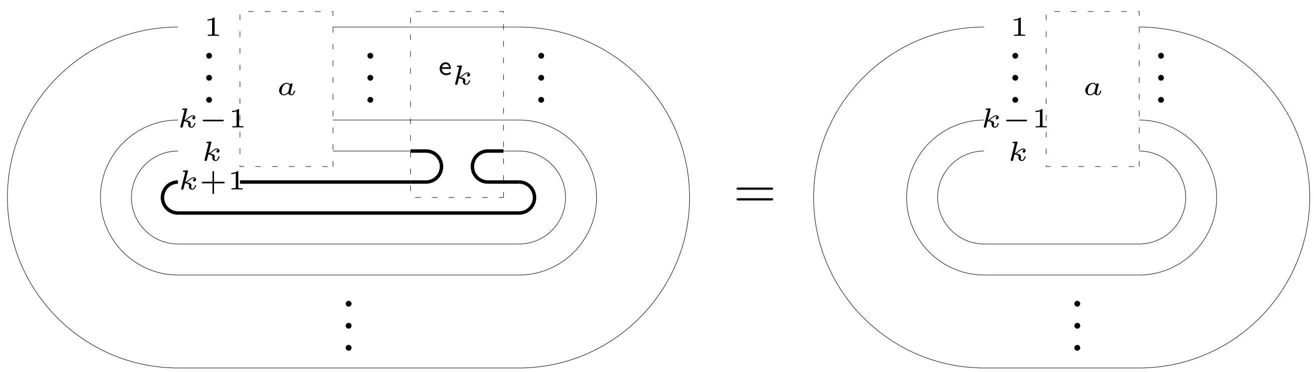


Use $\#_{\mathrm{loop}}(S)$ to denote the number of loops in the closure of S.

The family of functions tr_k defined on strand diagrams by

$$\mathrm{tr}_k(S) = q^{\#_{\mathrm{loop}}(S)}, \tag{8}$$

and extended linearly to $\mathfrak{TL}_k(q)$ is the Markov trace such that $\mathrm{tr}_1(1) = q$.



The Markov property allows recursive calculation.

Example 4.1.

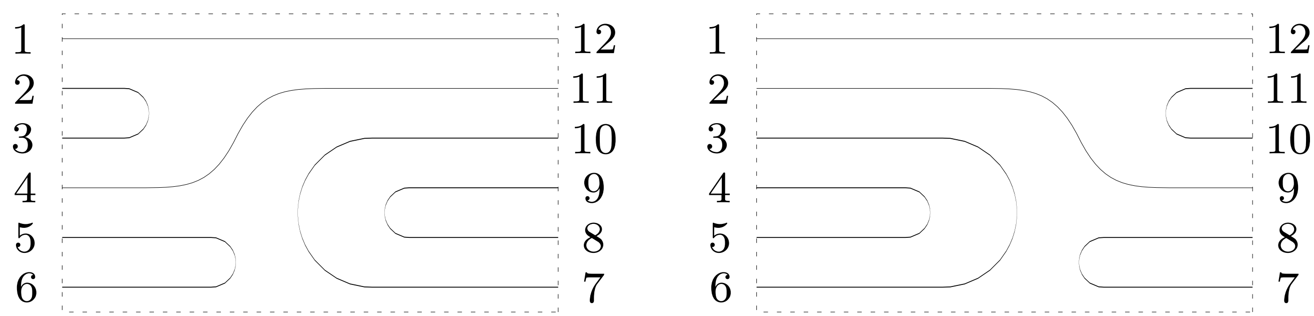
$$\begin{aligned} q^{\#_{\text{loop}}(\mathsf{e}_5\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5\mathsf{e}_4)} &= \operatorname{tr}_6(\mathsf{e}_5\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5\mathsf{e}_4) = \operatorname{tr}_6(\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5\mathsf{e}_4\mathsf{e}_5) \\ &= \operatorname{tr}_6(\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5) = \operatorname{tr}_5(\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3) \\ &= \operatorname{tr}_4(\mathsf{e}_2\mathsf{e}_3) = \operatorname{tr}_3(\mathsf{e}_2) = \operatorname{tr}_2(1) = q^2 \end{aligned}$$

Example 4.2. *Using (R3),* tr_{k+1} *can be recursively evaluated in terms of* tr_k *even when* e_k *does not appear.*

$$\operatorname{tr}_4(\mathsf{e}_1) = \operatorname{tr}_4(\mathsf{e}_1\mathsf{e}_2\mathsf{e}_1) = \operatorname{tr}_4(\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3\mathsf{e}_2\mathsf{e}_1) = \operatorname{tr}_3(\mathsf{e}_1\mathsf{e}_2\mathsf{e}_2\mathsf{e}_1)$$

A Transpose

Definition 7. *The* transpose *function* ${}^t \colon \mathfrak{TL}_n(q) \to \mathfrak{TL}_n(q)$ *is defined on the multiplicative basis by* ${\mathsf{e}_i}^t = \mathsf{e}_i$ *and extended to the whole algebra by* $(\mathsf{ef})^t = \mathsf{f}^t\mathsf{e}^t$ *and* $(\lambda\mathsf{e} + \gamma\mathsf{f})^t = \lambda\mathsf{e}^t + \gamma\mathsf{f}^t$*.*



Using ^{t}, the symmetric bilinear form

$$\langle\cdot,\cdot\rangle_n \colon \mathfrak{TL}_n(q) \times \mathfrak{TL}_n(q) \to \mathbb{C}(q) \tag{9}$$

$$(\mathsf{e},\mathsf{f}) \mapsto \mathrm{tr}_n(\mathsf{e}\mathsf{f}^t) \tag{10}$$

has the property that if e and f are the representation of arch configurations a and b, then $\langle \mathsf{e},\mathsf{f}\rangle = q^{c(a,b)}$, where $c(a,b)$ is the number of components in the meandric system with a as its upper configuration and b as its lower configuration.

This gives the expression

$$m_n(q) = \sum_{k=1}^{n} M_n^{(k)} q^k = \sum_{\mathsf{e},\mathsf{f}\in\mathcal{B}_1} \langle \mathsf{e},\mathsf{f}\rangle_n, \tag{11}$$

for the n-th meandric polynomial.

The Gram Matrix

$\langle\cdot,\cdot\rangle_n$ is summarized by its *Gram matrix*, $\mathbf{M}_n(q)$, a $C_n \times C_n$ matrix such that

$$\big(\mathbf{M}_n(q)\big)_{ij} = \langle \mathsf{a}_i, \mathsf{a}_j \rangle_n,$$

where $(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_{C_n})$ is an ordered basis of strand diagrams. This gives

$$m_n(q) = \sum_{i,j=1}^{C_n} \big(\mathbf{M}_n(q)\big)_{ij} = \mathrm{tr}(\mathbf{M}_n(q) \cdot \mathbf{J}_n) = {u_n}^t\, \mathbf{M}_n(q) u_n, \text{ and}$$

$$m_n(q^2) = \mathrm{tr}\left(\mathbf{M}_n(q)^2\right),$$

where u_n is the C_n dimensional column vector of 1's and $\mathbf{J}_n$ is the $C_n \times C_n$ matrix of 1's.

Example 4.3. *Using the basis* $\{\mathsf{e}_1, \mathsf{e}_2\mathsf{e}_1, \mathsf{e}_1\mathsf{e}_2, \mathsf{e}_2, 1\}$*, the Gram matrix of* $\langle\cdot,\cdot\rangle_3$ *is*

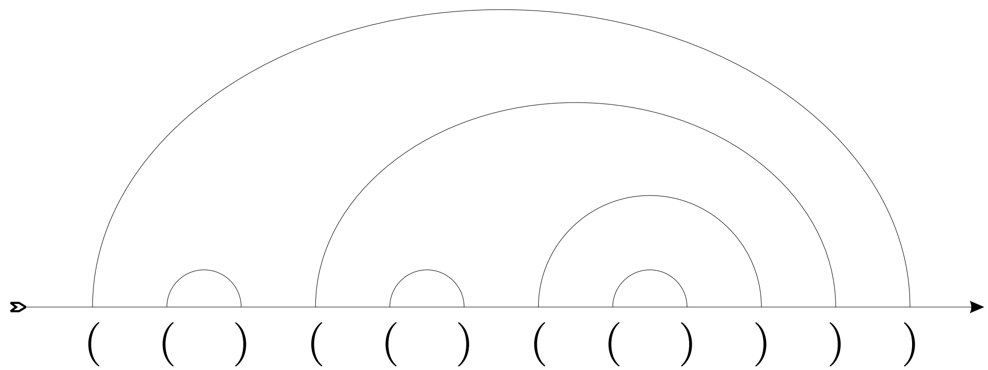
$$\mathbf{M}_3(q) = \begin{pmatrix} q^3 & q^2 & q^2 & q & q^2 \\ q^2 & q^3 & q & q^2 & q \\ q^2 & q & q^3 & q^2 & q \\ q & q^2 & q^2 & q^3 & q^2 \\ q^2 & q & q & q^2 & q^3 \end{pmatrix}.$$

So $m_3(q) = \mathrm{tr}(\mathbf{M}_3(q)\mathbf{J}_3) = 8q + 12q^2 + 5q^3$*,*
and $M_3 = [q]m_3(q) = 8$*.*

5. Combinatorial Words

The Encoding

Arch configurations have a natural encoding as balanced parentheses.



By separately encoding the upper and lower configurations of meandric systems as words in the Dyck language, and using the rule,

$$\begin{matrix}(\\(\end{matrix}\to O \qquad \begin{matrix})\\)\end{matrix}\to C \qquad \begin{matrix}(\\)\end{matrix}\to U \qquad \begin{matrix})\\(\end{matrix}\to D,$$

this extends to meandric systems.

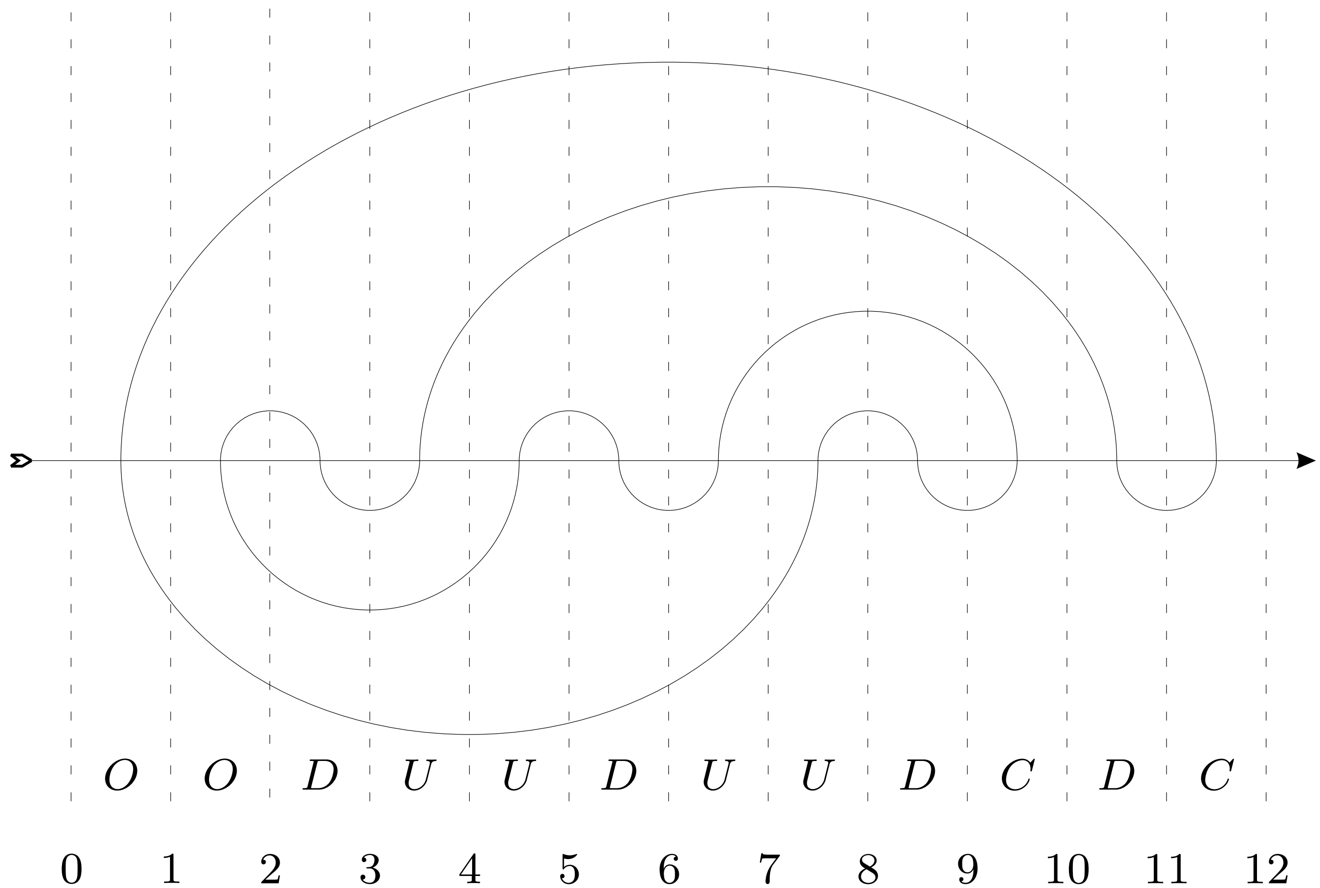


Figure 1: The meander ‘*OODUUDUUDCDC*’

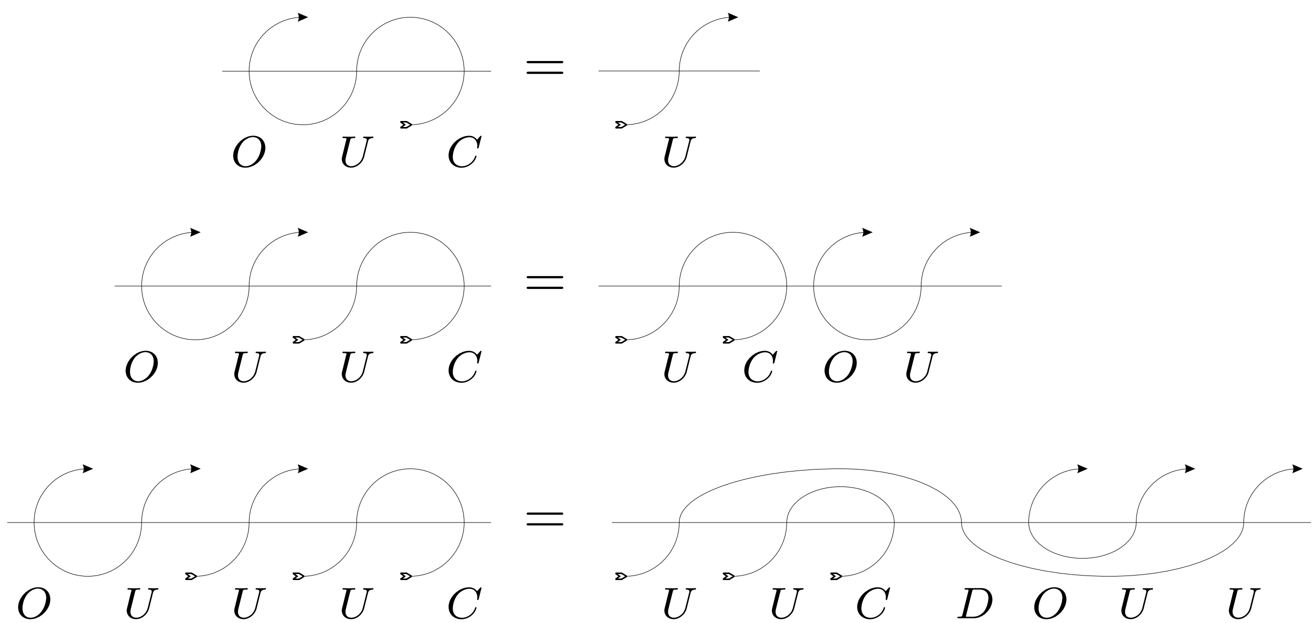
The Language of Meanders

The language of meanders is closed under the substitutions

$$
\begin{aligned}
OUC &\leftrightarrow U & ODC &\leftrightarrow D \\
OUUC &\leftrightarrow UCOU & ODDC &\leftrightarrow DCOD \\
OUUUC &\leftrightarrow UUCDOUU & ODDDC &\leftrightarrow DDCUODD \\
&\vdots & &\vdots \\
OU^{i+2}C &\leftrightarrow U^{i+1}CD^iOU^{i+1} & OD^{i+2}C &\leftrightarrow D^{i+1}CU^iOD^{i+1}
\end{aligned}
$$

and

$$
UD \to \epsilon \qquad DU \to \epsilon,
$$


O U C = U
O U U C = U C O U
O U U U C = U U C D O U U