The Rubik’s Cube
The Moves

Each face of the cube can be rotated.

The result is a permutation of the stickers and physical pieces (cubies) that make up the cube.
The faces of the cube are denoted: (F)ront, (B)ack, (L)eft, (R)ight, (U)p, and (D)own

The cube group, $G_{cube}$, is the permutation group generated by the actions of the six face turns on the stickers.
Only Five Generators Required

Figure 1: A commuting diagram
Counting the States

We consider first a larger permutation group, $G$ of all permutations obtainable by taking apart the cube and putting it back together.

$$G \cong G_{\text{corner}} \oplus G_{\text{edge}}$$

$G_{\text{corner}}$ and $G_{\text{edge}}$ are wreath products.

$$G_{\text{corner}} \cong S_8[A_3] \quad G_{\text{edge}} \cong S_{12}[S_2]$$

The order of $G$ is thus:

$$|G| = |G_{\text{corner}}| \cdot |G_{\text{edge}}| = 8! \times 3^8 \times 12! \times 2^{12}$$
The Order of $G_{cube}$

We show that $G_{cube}$ has index 12 in $G$ and thus:

$$|G_{cube}| = 12! \times 8! \times 2^{10} \times 3^7$$

- 18 cubie positions determine the remaining 2
- 11 edges orientations determine the twelfth
- 7 corner orientations determine the eighth
An Alternate Colouring

The sum of the orientations of the corners is always zero.
Generating the Edge Group

Two related commutators.

The restriction to the edge group.
Another commutator gives us a three cycle to position corners. Combining this with a related commutator lets us orient the corners.
Diameter of the Cayley graph

For the quarter turn metric. An edge when states differ by an element of \( \{L, R, F, B, U, D, L', R', F', B', U', D'\} \).

- Lower bound of 24 for the super-flip

![Cube Diagram](image1)

![Graph Diagram](image2)

Figure 2: \( R'U^2BL'FU'BDFUD'L^2F'RB'DF'U'B'UD' \)
• Upper bound of 42 using Kloosterman’s modification of Thistlethwaite’s algorithm.

\[ G_0 = \langle L, R, F, B, U, D \rangle \]
\[ G_1 = \langle L, R, F, B, U^2, D^2 \rangle \]
\[ G_2 = \langle L, R, F^2, B^2, U^2, D^2 \rangle \]
\[ G_3 = \langle L^2, R^2, F^2, B^2, U^2, D^2 \rangle \]
See http://web.idirect.com/ cubeman/dotcs.txt for the face turn metric.

http://www.geocities.com/jaapsch/puzzles/cayley.htm puts the state

\[ U^2 D^2 LF^2 U' D R^2 B U' D' R L F^2 R U D' R' L U F' B' \]

at distance 26q.