

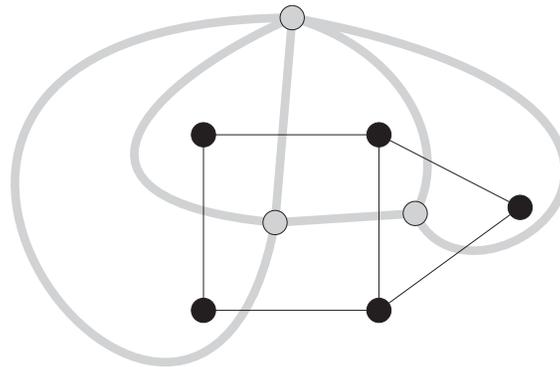
# Two Conjectures on Enumerative Properties of Rooted Maps

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# Maps

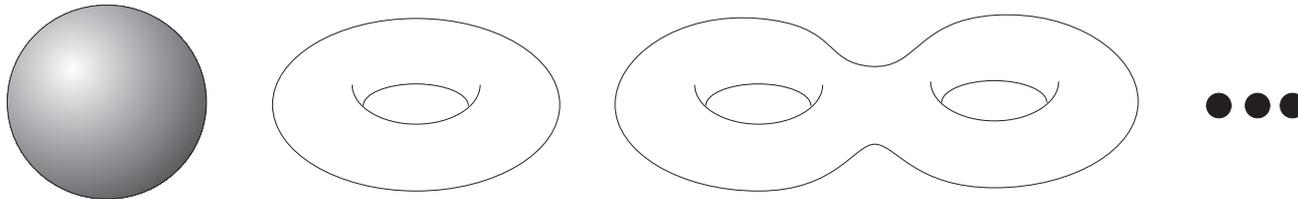
A *map* is a 2-cell embedding of a graph (which may have loops and multiple edges) in a surface. Deletion of the graph decomposes the surface into regions homeomorphic to the open disc.



These discs are the faces of the map. This produces a dual incidence structure, with faces taking the role of vertices.

## Surfaces

A surface is a compact, connected, 2-manifold. Every surface is homeomorphic to a sphere, a sphere with a finite number of handles, or a sphere with a finite number of cross-caps.



The orientable surfaces embed in 3-space. Non-orientable surfaces are representable schematically.

# Ribbon Graphs

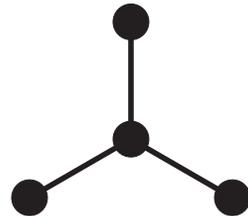
Throwing away everything except a neighbourhood of the edges leaves a *ribbon graph*. The map is recoverable by stitching disks along boundary components.



The ribbon graph representation gives each edge 4-corners.

## Equivalence of Maps

Two maps are equivalent if one can be transformed into the other via homeomorphism. This corresponds to the ribbon graphs of the maps inducing the same incidence structure on edge corners.



A map may have non-trivial automorphisms.

## Rooted Maps

It is useful to consider the orbits of edge-corners under the action of a map's automorphism group. An orbit can be specified by marking a point of incidence between a vertex and a face, and establishing a sense of clockwise about that point.



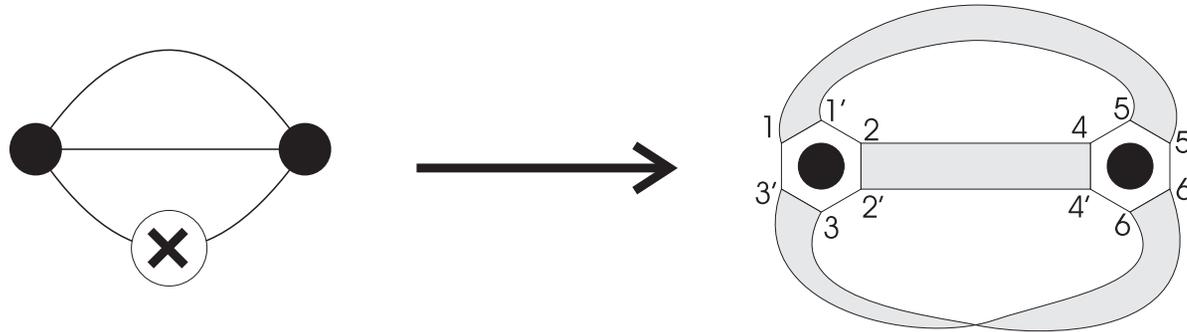
The marked object is a *rooted map*.

## The $b$ -Conjecture

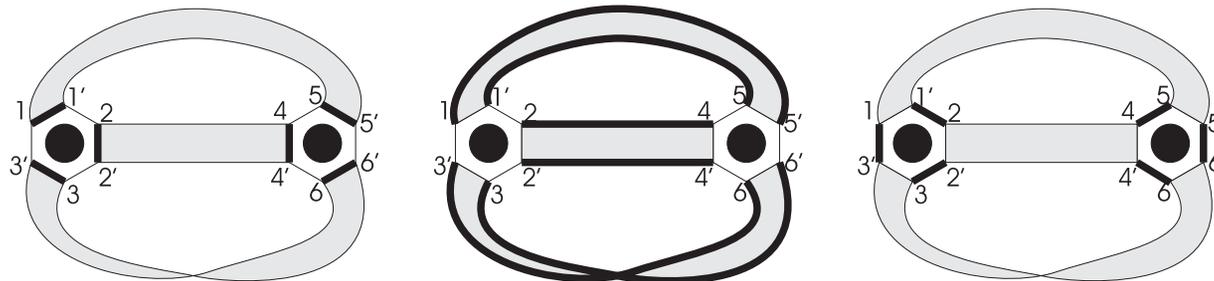
The  $b$ -conjecture arises from similarities between the class of all rooted maps and the class of rooted maps on orientable surfaces.

An algebraic encoding relates the number of decorated pre-hypermaps in each class to connection coefficients in related algebras. The  $b$ -conjecture seeks a combinatorial interpretation for a common generalization that is understood only on the level of the algebra.

# Endcoding Maps

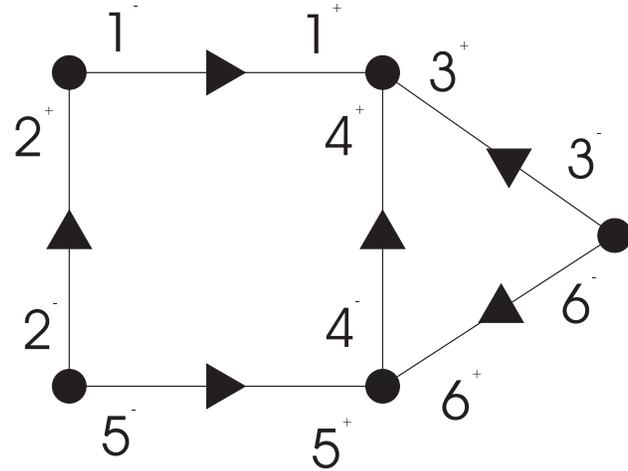
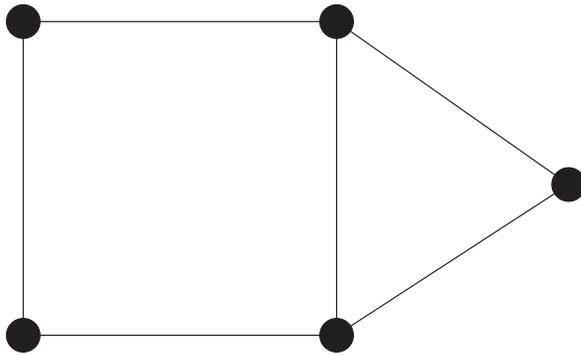


The ribbon graph gives 3 perfect matchings on edge corners.



Faces, vertices, and hyper-edges correspond to cycles of the unions of matchings taken in pairs.

## Encoding Orientable Maps



$$\epsilon = (1^+ 1^-)(2^+ 2^-)(3^+ 3^-)(4^+ 4^-)(5^+ 5^-)(6^+ 6^-)$$

$$\nu = (1^- 2^+)(1^+ 3^+ 4^+)(2^- 5^-)(3^- 6^-)(4^- 6^+ 5^+)$$

$$\varphi = \nu\epsilon = (1^- 2^- 5^+ 4^+)(1^+ 3^- 6^+ 5^- 2^+)(3^+ 4^- 6^-)$$

## Two Algebras

Decorated pre-hypermaps are counted by connection coefficients in the double-coset algebra of the hyper-octahedral group,

$$\mathcal{H}(\mathfrak{S}_{2n}, \mathfrak{S}_n[\mathfrak{S}_2]).$$

Oriented decorated pre-hypermaps are counted by connection coefficients in the class algebra of the symmetric group,

$$Z_{\mathbb{C}\mathfrak{S}_n}.$$

## Two Generating Series

For locally orientable maps we get:

$$H(\mathbf{p}(\mathbf{x}), \mathbf{p}(\mathbf{y}), \mathbf{p}(\mathbf{z}), 0) = t \frac{\partial}{\partial t} \log \sum_{\theta \in \mathcal{P}} H_{\theta} t^{|\theta|} s_{\theta}(\mathbf{x}) s_{\theta}(\mathbf{y}) s_{\theta}(\mathbf{z}) \Big|_{t=1},$$

For orientable maps we get:

$$H(\mathbf{p}(\mathbf{x}), \mathbf{p}(\mathbf{y}), \mathbf{p}(\mathbf{z}), 1) = 2t \frac{\partial}{\partial t} \log \sum_{\theta \in \mathcal{P}} \frac{1}{H_{2\theta}} t^{|\theta|} Z_{\theta}(\mathbf{x}) Z_{\theta}(\mathbf{y}) Z_{\theta}(\mathbf{z}) \Big|_{t=1},$$

With  $H_{\theta}$  the product of the hook-lengths of the partition  $\theta$ .

## A Common Generalization

$$H(\mathbf{p}(\mathbf{x}), \mathbf{p}(\mathbf{y}), \mathbf{p}(\mathbf{z}), b) = (1 + b)t \frac{\partial}{\partial t} \log \sum_{\theta \in \mathcal{P}} \frac{t^{|\theta|}}{\langle J_\theta, J_\theta \rangle_{1+b}} J_\theta(\mathbf{x}, 1 + b) J_\theta(\mathbf{y}, 1 + b) J_\theta(\mathbf{z}, 1 + b) \Big|_{t=1}.$$

Here  $J_\theta(\mathbf{x}, \alpha)$  are the result of applying Gram-Schmidt to monomial symmetric functions with respect to

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda, \mu} \frac{|\lambda|!}{|C_\lambda|} \alpha^{\ell(\lambda)}.$$

## $b$ Polynomials

Numerical evidence shows that the coefficients in  $H(\mathbf{x}, \mathbf{y}, \mathbf{z}, b)$  are polynomials in  $b$  with non-negative integer coefficients.

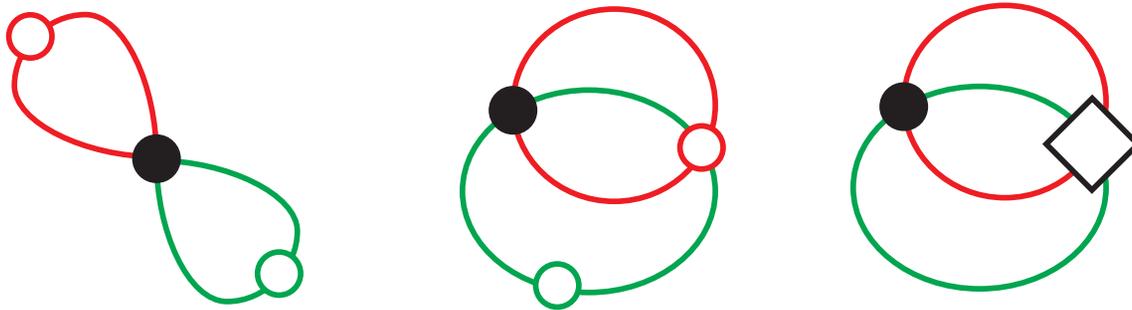
The  $b$ -conjecture states that there exists a natural invariant of rooted hyper-maps with respect to which  $H(\mathbf{x}, \mathbf{y}, \mathbf{z}, b)$  is their generating series.

The invariant takes the value 0 for orientable maps, and is bounded between 1 and  $n$  for a rooted map on the sphere with  $n$  cross-caps. It is not an invariant of maps.

## Example

Considering one-faced maps with one vertex and two edges.

$$[x_4 y_4 z_2^2] H(\mathbf{x}, \mathbf{y}, \mathbf{z}, b) = 1 + b + 3b^2$$



The right map contributes 1, the two rootings of each of the non-orientable map must contribute a total of  $b + 3b^2$ .

## A Possible Invariant

D. Brown and D. Jackson give an algorithm for computing an invariant of rooted maps. It agrees with the conjectured invariant for maps with a small number of edges.

The polynomials  $J_\theta(\mathbf{x}, \alpha)$  are not understood well enough to attempt a complete verification.

## One-Faced Monopoles

The algorithm suggests that every map has, as a minor, a one-faced monopole on the same surface for which the invariant takes the same value.

It might be possible to verify that the computed invariant agrees with the conjectured invariant on this class.

## Is This Useful?

Interpreting Schur functions and zonal polynomials as spherical functions gives an integral expression for an evaluation of  $H(\mathbf{x}, \mathbf{y}, \mathbf{z}, b)$  with

$$y = y_1 = y_2 = y_3 = \dots$$

$$0 = z_1 = z_3 = z_4 = \dots$$

This evaluation retains full information for one-faced maps, and is tractable for one-vertex maps.

## What's Still Needed

Computation of the invariant of Brown and Jackson does not remain within the class of one-faced monopoles. Verficiation requires a calculus on rooted one-faced monopoles, and a way to relate the calculus to the integral expressions of the map generating series.

# The $q$ -Conjecture

## The Facts

Let  $M(u^2, x, y, z)$  be the generating series for rooted oriented maps with respect to genus, number of vertices, number of faces, and number of edges. If  $M_4(u^2, x, y, z)$  is the restriction of this series to rooted quadrangulations then T. Visentin and D. Jackson showed

$$M(u^2, x, y, z) = \text{bis}_{u=0} M_4((2u)^2, x + u, x, yz^2),$$

where  $\text{bis}_{u=0} f(u) = \frac{1}{2}(f(u) + f(-u))$  denotes the even bisection of  $f$  with respect to  $u$ .

## The Conjecture

The  $q$ -conjecture asserts that the previous equation can be explained in terms of the existence of a natural map

$$\Xi: \mathcal{Q} \rightarrow \mathcal{M}$$

from the set of rooted quadrangulations to the set of all rooted oriented maps.

An element of  $\mathcal{M}$  with  $v$  vertices,  $f$  faces, and  $e$  edges, and genus  $g$  should have  $2^{2g+v-1}$  preimages. In particular, it should have  $2^{2g} \binom{v}{2i}$  pre-images of genus  $g + i$  for each  $i$  in the range  $0 \leq i \leq \lfloor \frac{v}{2} \rfloor$ .

## Starting With Edges

The factor  $2^{2g+v-1}$  is suggestive. If the edge set of a map  $G$  is partitioned as

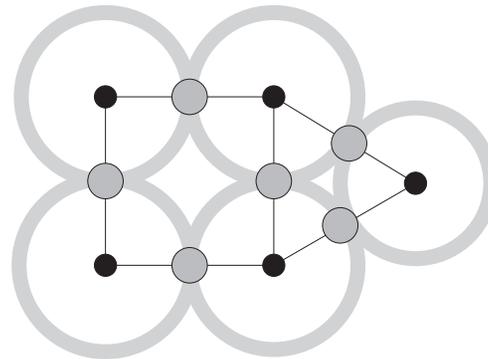
$$E = S \dot{\cup} D \dot{\cup} B$$

where  $S$  is the edge set of a spanning tree of  $G$ , and  $D$  is the edge set of a dual-spanning tree, then  $G$  has one preimage under  $\Xi$  for each subset of  $S \cup B$ .

By decorating each element of  $\mathcal{M}$  by marking an appropriate subset of edges, it may be possible to describe  $\Xi$  as a bijection.

## The Medial Construction

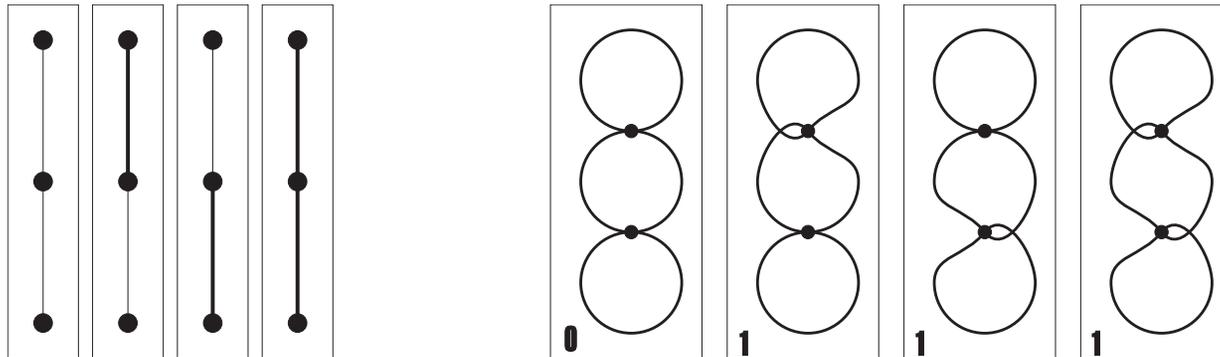
The medial construction gives a bijection between rooted maps and 2-colourable rooted quadrangulations on the same surface. A map and its dual have the same image, but with different rootings.



It is appealing to consider only  $\Xi$  whose restrictions correspond to the medial construction.

# Trees

A tree must have one image under  $\Xi^{-1}$  for each subset of its edges.  
A candidate for  $\Xi^{-1}$  with the correct cardinality but incorrect genus distribution begins with the medial construction, and introduces a twist at vertices corresponding to marked edges.



## One-Faced Monopoles

The restriction of  $\Xi^{-1}$  to one-faced monopoles looks interesting for numerical reasons. There are

$$\begin{aligned}
 [N] \frac{(2n)!}{2^n n!} \sum_{k=1}^{n+1} \binom{N}{k} \binom{n}{k-1} 2^{k-1} &= \frac{(2n)!}{2^n n!} \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} (-2)^k \\
 &= \frac{(2n)!}{2^n (n+1)!}
 \end{aligned}$$

such rooted-oriented monopoles with  $n$  edges when  $n$  is even.

This is notable for the fact that, where  $C_n$  is the  $n$ 'th Catalan number, these monopoles have a total of  $n!C_n$  images under  $\Xi^{-1}$ , and suggests the possible involvement of edge-labelled trees at an intermediate stage.

## TDT-rooted Maps

A *TDT-rooted* map is a rooted map whose edge set has been partitioned as  $E = S \dot{\cup} D \dot{\cup} B$ , with  $S$ ,  $D$ , and  $B$  as before. For some of the classes on maps discussed in this talk, TDT-rooted maps correspond to undecorated maps. This is an extension of the idea of a tree-rooted map, and displays the symmetry between a map and its dual.

## Construction TDT-Rooted Maps

The edge sets  $S$ ,  $D$ , and  $B$  induce rooted minors that are respectively, a tree, the dual of a tree, and a one-faced monopole. A map can be recovered from these three induced maps by specifying an interleaving of edge-ends.

The vertex of the one-faced monopole is first replaced by the spanning tree. The single face of the resulting map can then be replaced by the dual-spanning tree.

## Counting TDT-Rooted Maps

For  $S$ ,  $D$ , and  $B$  with

$$|S| = v - 1, \quad |D| = f - 1, \quad \text{and} \quad |B| = 2g,$$

the interleaving can take place in

$$\binom{4g + 2v - 2}{2v - 2} \binom{4g + 2v + 2f - 4}{2f - 2}$$

ways to produce a map.

## A Hidden Symmetry?

The maps  $S$ ,  $D$ , and  $B$  can be constructed in

$$C_{v-1}, \quad C_{f-1}, \quad \text{and} \quad \frac{(2g)!}{2^{2g}} C_{2g}$$

ways, respectively, in the orientable case. This gives a total of

$$\frac{(2e)!}{2^{2g}(2g+1)!v!(v-1)!f!(f-1)!}$$

TDT-rooted maps.

The construction, and the tree-like structure of one-faced monopoles suggests that there may be a hidden symmetry, under which  $S$  and  $B$  may be interchanged in some meaningful way.

## TDT-Rooted Quadrangulations

A TDT-rooted quadrangulation of genus  $g$  with  $n$  faces, has

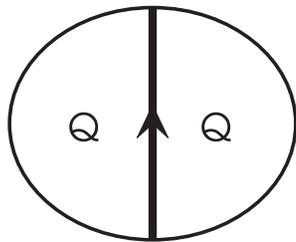
$$|B| = 2g, \quad |S| = -2g + n + 1, \quad \text{and} \quad |D| = n - 1.$$

The map remaining when  $D$  is deleted has a single face of degree  $2|S \cup B|$ .

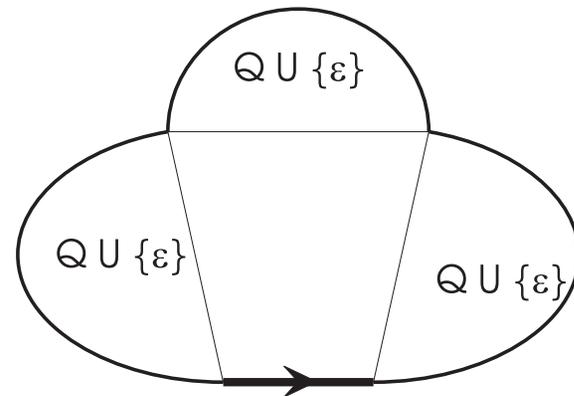
With this restriction,  $S$  and  $B$  can be selected freely. It remains only to consider the number of ways to quadrangulate the single face.

## Rooted Quadrangulations of a Face

Let  $Q(x)$  be the generating series for the set of all quadrangulations rooted on the exterior face with respect to number of interior faces.



Rootings not on the exterior face are generated by  $(Q(x))^2$ .



Rootings on the exterior face are generated by

$$Q(x) = x(1 + Q(x))^3.$$

## Counting TDT-Rooted Quadrangulations

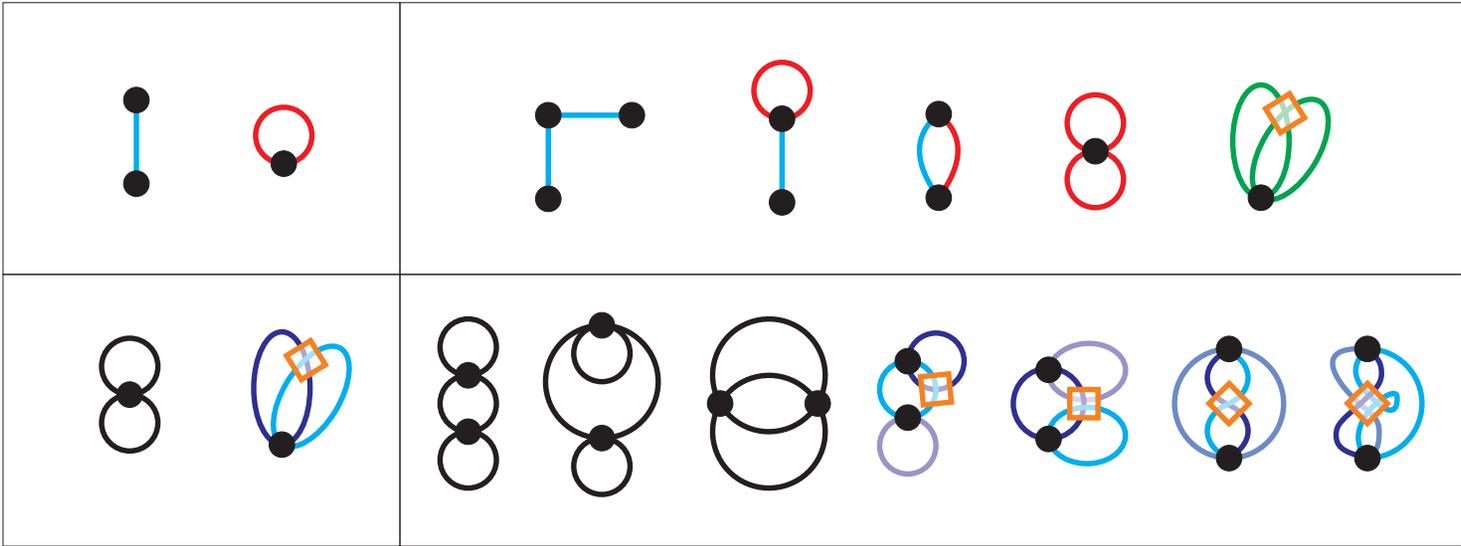
Multiplying the number of choices at each step of the construction shows that there are

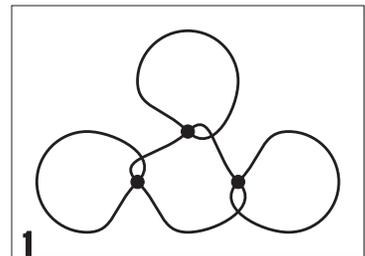
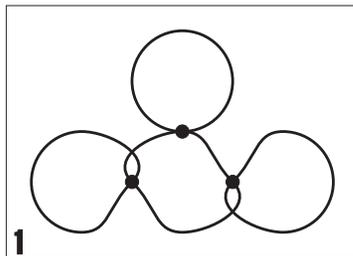
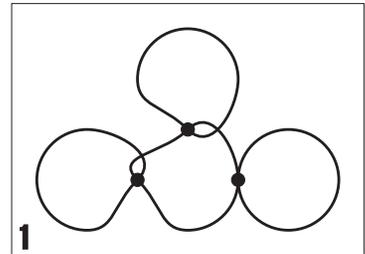
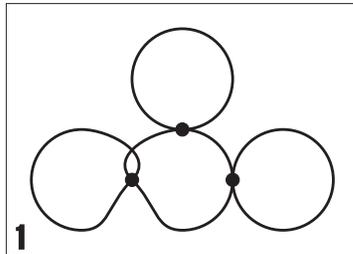
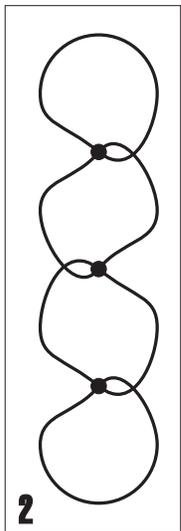
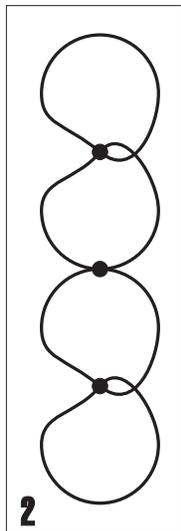
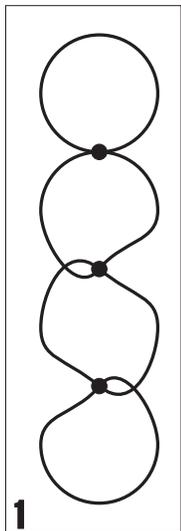
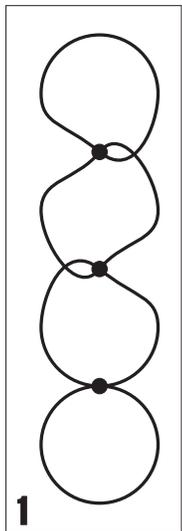
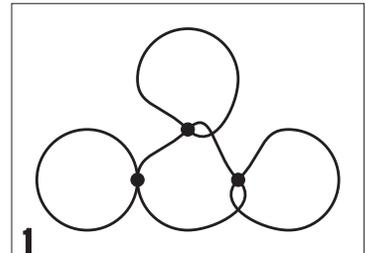
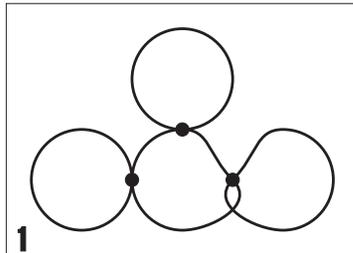
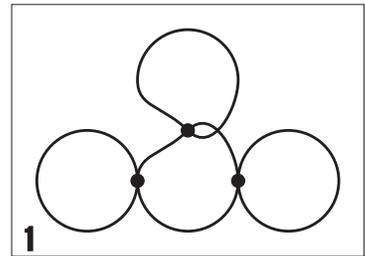
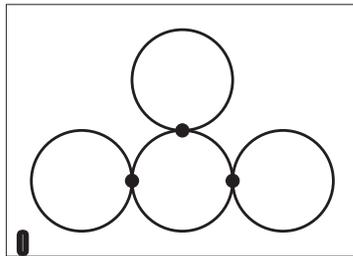
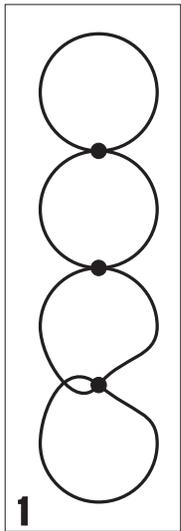
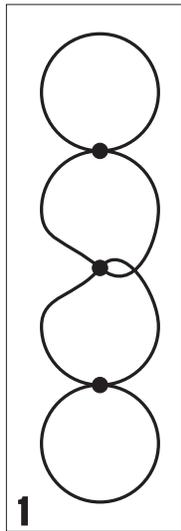
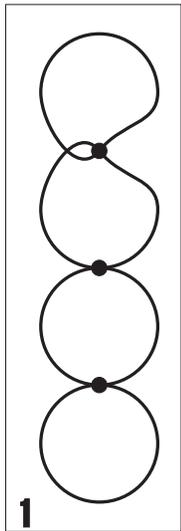
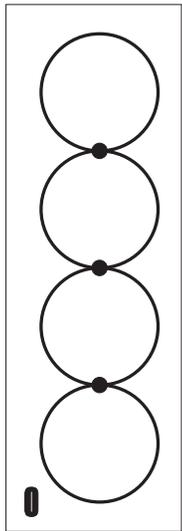
$$\begin{aligned} & \frac{(2g)!}{2^{2g}} C_{2g} C_{n+1-2g} \binom{2n+2}{4g} \left[ \frac{1}{n} \binom{3n}{n-1} + \frac{2}{n} \binom{3n}{n-2} \right] \\ & = \frac{4 \cdot (3n)!}{2^{2g} (2g+1)! v! (v-1)! (n-1)!} \end{aligned}$$

TDT-rooted quadrangulations.

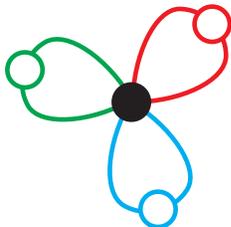
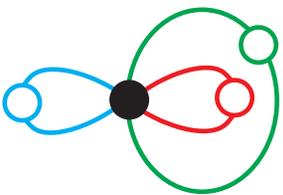
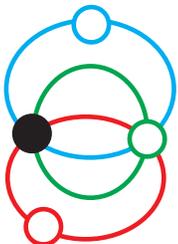
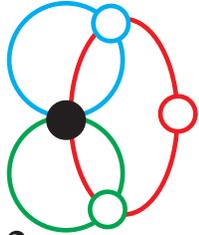
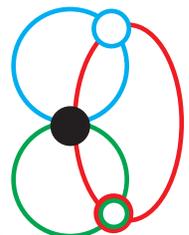
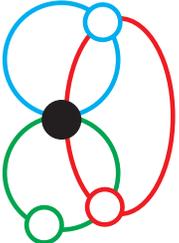
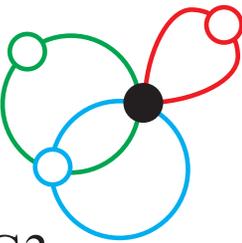
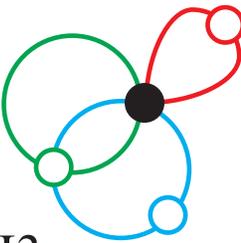
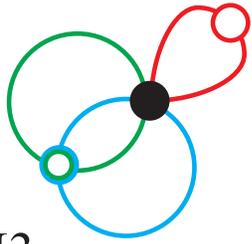
# Extra Slides

# Extra Figures





# 3-Edged Non-Orientable One-Faced Monopoles

 <p>A3</p>	 <p>B3</p>	 <p>C3</p>	 <p>D3</p>
 <p>E3</p>	 <p>F3</p>	 <p>G3</p>	 <p>H3</p>
 <p>I3</p>			

