Approaches to the Enumerative Theory of Meanders

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September 29, 2003

Contents

List of Figures

Chapter 1

Introduction

Meanders are combinatorial objects with a topological flavour, encapsulating properties of the interplay between planarity and connectedness. They correspond to the systems formed by the intersections of two curves in the plane, with equivalence up to homeomorphism within the plane. They arise in other guises in polymer physics, algebraic geometry, and the study of planar algebras, especially the Temperley-Lieb algebra. For applications of meanders, the reader is referred to [1, 4, 12]. Associated with each meander, is a crossing number. The focus of this essay is the problem of enumerating inequivalent closed meanders with respect to their crossing numbers.

The problem of enumerating meanders has a long history, with interest dating back at least to work by Poincaré on differential geometry, though the modern study appears to have been inspired by Arnol′d in [1]. Also considered, will be the related problem of enumerating semi-meanders. Emphasis will be placed on exact results, but asymptotic approximations will also be described.

This essay presents several constructions that can be used to express meandric numbers in terms of other combinatorial objects. The resulting expressions are elegant, but computationally intractable to date. That expressions for meandric numbers can be derived through such diverse constructions, illustrates a sublte interrelation between seemingly unrelated areas of mathematics.

1.1 Definitions

We begin by defining the objects of study. Three varieties of meanders will be of particular interest in this essay, the most natural to describe being the open meander, which we introduce informally through a geographical analogy before defining it formally.

Informally, an open meander may be represented as the configuration formed by a river and a road. The river approaches the road from the northwest, meanders around under an east-west road, and then continues off to the east. Both the river and the road can be considered to be infinite. See Figure 1.1. It is from this geographical context that the name meander is derived. The imagery of this description encapsulates several of the key features of meanders that need to be formalized.

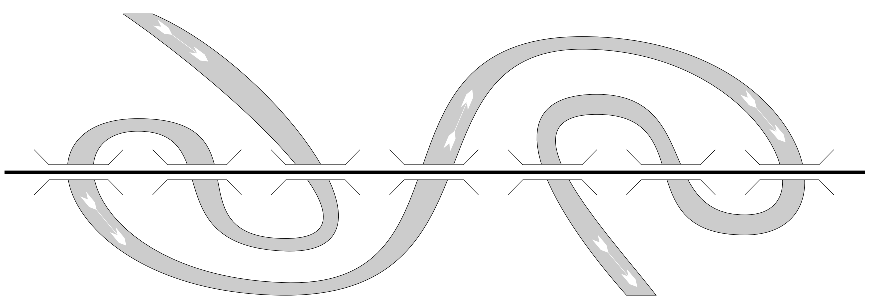


Figure 1.1: An open meander represented as a river and a road.

A meander consists of two distinguished simple planar curves; in the case of the geographical analogy these are the river and the road. The two curves cross a finite number of times and these crossings account for all of the intersections between the curves. We are interested in meanders only up to equivalence under homeomorphism within the plane and impose an orientation on the river to prevent unwanted equivalences arising from rotating a configuration to swap the ends of the river or road. Combining these features, we can now define an open meander.

Definition 1.1.1 (Open Meander). *An* open meander *is a configuration consisting of an oriented simple curve and a line in the plane, that cross* *a finite number of times and intersect only transversally. Two open meanders are equivalent if there is a homeomorphism of the plane that maps one meander to the other.*

A natural parameter of an open meander is its order. Given a meandric configuration, the *order* of the meander is the number of crossings between the two curves. Figure 1.2 gives three examples of open meanders and indicates their orders.

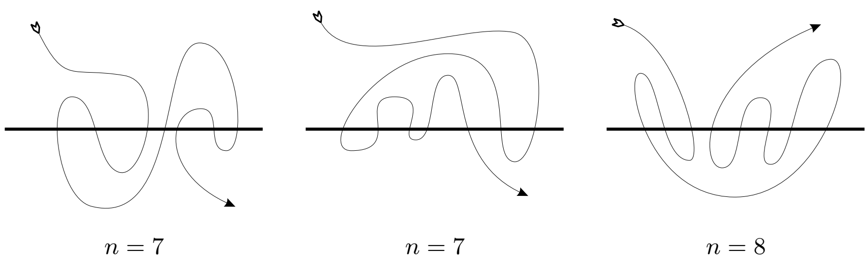


Figure 1.2: Examples of open meanders and their orders, denoted by n.

It is worth noting that in the definition of an open meander, the line and curve play symmetric roles. Since a line and a simple curve are homeomorphic, and we are only interested in defining meanders up to homeomorphism, their roles can be reversed by imposing an orientation on the line instead of the curve. A homeomorphism of the plane can then be used to convert this configuration to the familiar form. In light of this equivalence between the curves, it will often be convenient to orient the line instead of the curve when defining other varieties of meanders.

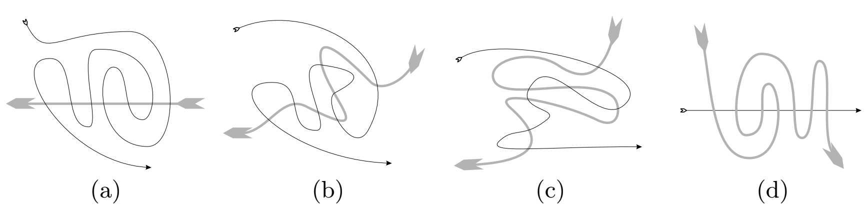


Figure 1.3: The duality between the line and curve in an open meander

This duality is illustrated in Figure 1.3. Beginning with an open meander we orient the line (a). The curve is straightened by applying a homeomorphism of the plane (b-d). Straightening the curve, curves the line. By ignoring the final orientation on the line, we are left with a new open meander (d). By our choice in orientations, applying the transformation a second time has the result of rotating the configuration by a half turn in the plane. The process can be made self inverse by reflecting the final configuration across the line.

For the purpose of enumeration a more convenient class of objects of study is the class of closed meanders, and it is customary to refer to these simply as meanders.

Definition 1.1.2 (Closed Meander). *A* closed meander *is a planar configuration consisting of a simple closed curve and an oriented line, that cross finitely many times and intersect only transversally. Two meanders are equivalent if there exists a homeomorphism of the plane that maps one to the other.*

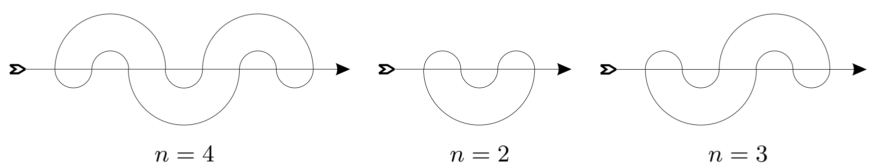


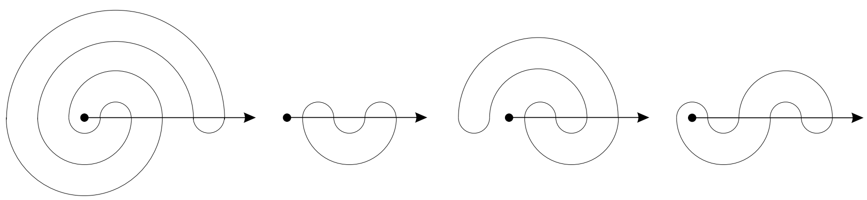
Figure 1.4: Examples of closed meanders, and their orders, denoted by n.

The order of a closed meander is defined as the number of pairs of intersections between the closed curve and the line. Since the two curves intersect an even number of times, the order of a closed meander is an integer. Examples of closed meanders are provided in Figure 1.4.

A third variety of meanders is obtained by replacing the line in the definition of a close meander, with a ray. If we think of the ray as a river with a source, and think of the curve as a road, this corresponds to a road that wraps around the source of the river.

Definition 1.1.3 (Semi-meander). *A* semi-meander *is a planar configuration consisting of a simple closed curve and a ray, that cross finitely many* *times and intersect only transversally. Two semi-meanders are equivalent if there exists a homeomorphism of the plane that maps one to the other.*

As with open meanders, the number of intersections between the two curves of a semi-meandric configuration may be odd, and the order of a semi-meandric configuration is the number of intersections between the two curves. In addition to order, semi-meanders have a second natural parameter of enumerative importance. The *winding number* of a semi-meander is the minimum, over all equivalent configurations, of the number of times the closed curve crosses the extension of the ray to a line. Examples of semi-meanders are provided in Figure 1.5. Notice that the middle two examples are equivalent as semi-meanders.



$n = 5$, $w = 3$ $n = 4$, $w = 0$ $n = 4$, $w = 0$ $n = 5$, $w = 1$

Figure 1.5: Examples of semi-meanders, their orders, n, and their winding numbers, w.

The apparent inconsistency in the of definition of the order of a closed meander and the order of a semi-meander is justified in Section 2.2.1, where both classes are viewed in terms of arch configurations, which we now introduce.

Definition 1.1.4 (Arch Configuration). *An* arch configuration *is a planar configuration consisting of pairwise non-intersecting semicircular arches lying on the same side of an oriented line, arranged such that the feet of the arches are equally spaced along the line.*

The *order* of an arch configuration is the number of arches it contains. An arch configuration consisting of concentric arcs is called a *rainbow configuration.* Such configurations are used in representing semi-meanders. Figure 1.6 illustrates some examples of arch configurations. The final example is the rainbow configuration of order 5.

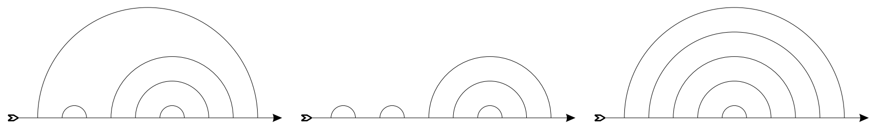


Figure 1.6: Examples of arch configurations of order 5.

Arch configurations play an essential role in the enumerative theory of meanders. They are used to obtain the canonical representatives of both meanders and semi-meanders that are used in subsequent constructions. They also have a natural link to the Temperley-Lieb algebra, which is discussed in Chapter 5.

The principal enumerative problem associated with meanders is to determine the number of inequivalent meandric configurations of various forms with respect to order. We introduce notation to represent these numbers for the varieties of meanders already introduced. The numbers of inequivalent open, closed, and semi-meanders of order n are denoted by m_n, M_n, and $\overline{M}_n$, respectively, and are called the n-th *open meandric number*, (*closed*) *meandric number*, and *semi-meandric number*, respectively. We let C_n denote the number of arch configurations of order n.

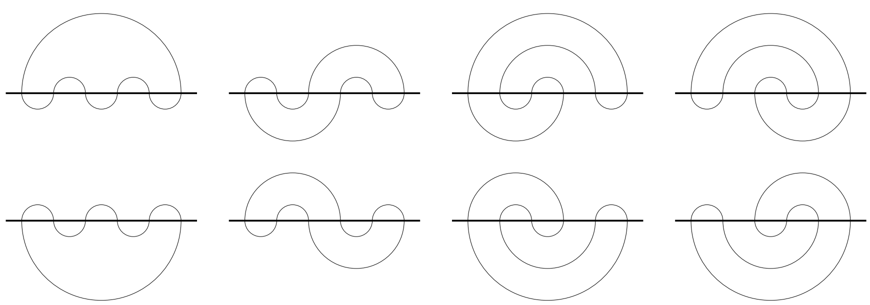


Figure 1.7: The eight inequivalent meandric configurations of order three.

As an example, Figure 1.7 gives all of the non-equivalent meandric configurations of order three, the lines each being understood to be oriented from left to right. So $M_3 = 8$. The first few meandric numbers are listed in Table A.1.

1.2 Enumerative Strategies

To date, there is not a complete solution to the meander problem. Several constructions can be used to rephrase it in different forms, but none of these approaches has yet succeeded in making the problem more tractable.

Chapter 2 presents some elementary approaches to the problem of enumerating meanders. Open meanders are related to closed meanders. Meanders and semi-meanders are then discussed in terms of arch configurations. This leads to a definition of meandric and semi-meandric systems, and in turn yields weak upper and lower bounds for the number of meanders of a given order. Several automorphisms on the class of meanders are described and used to derive congruences satisfied by the meandric numbers. The chapter concludes with a description of an algorithm for exhaustive enumeration of meanders and semi-meanders.

Each of the remaining chapters deals with a more advanced approach to the problem, outlining the construction and mathematical tools involved, and summarizing the conclusions that can be drawn from the approach.

Chapter 3 deals with representing meanders as permutations and then develops some properties of the symmetric group in order to express meandric and semi-meandric numbers in terms of characters of the symmetric group. These expressions provide an effective method for describing meanders in purely combinatorial terms, and a bridge to the theory of symmetric functions.

Chapter 4 presents an expression for meandric numbers in terms of matrix integrals. A meandric configuration is generalized to a map with multiple roads, multiple rivers, and arbitrary genus. Counting these modified objects with respect to number of rivers, roads, and genus is then phrased as a problem of colouring maps. This map colouring problem is then expressed as a matrix integral. Results about meanders can be obtained by taking appropriate limits. To date, there are no known techniques for evaluating the resulting integrals, but the construction rephrases the meander problem in analytic terms.

Chapter 5 introduces the Temperley-Lieb algebra. The meander problem is reduced to one of evaluating the Gram matrix of a symmetric bilinear form on the algebra with respect to a particular basis. The chapter develops some properties of the Temperley-Lieb algebra and relates them to the evaluation of this form. This construction potentially opens the meander problem to the tools of linear algebra.

Chapter 6 considers approaches to the meander problem that involve taking cross-sections of meanders. Each intersection between the curves is treated as a letter in a combinatorial word. This interpretation presents the meander problem in terms of the theory of formal languages and leads to a description of the technique that, to date, has been most effective at exhaustively enumerating meanders.

Chapter 2

Elementary Approaches

2.1 Relating Open and Closed Meanders

We first justify the exclusion of open meanders from subsequent discussion by noting their relationship to closed meanders, which are more easily analyzed, in a large part because of their representability by arch configurations.

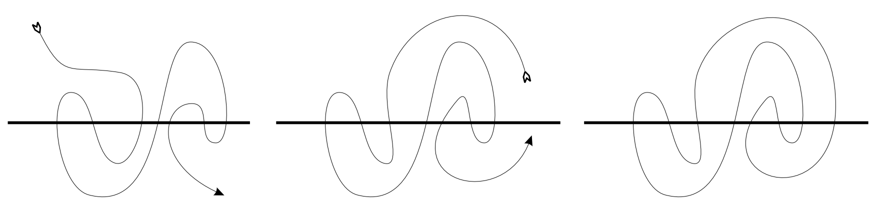


Figure 2.1: Closed meanders and odd order open meanders

Figure 2.1 illustrates the bijective correspondence between closed meanders and open meanders of odd order. A unique closed meander can be obtained from every open meander of odd order by adding an intersection between the curves at the right of the configuration. The two free ends of the curve are stretched around the rest of the configuration and made to meet at this point. The process is easily reversed by breaking the closed curve of a closed meander at its rightmost crossing with the line to obtain two free ends which are extended to infinity. This construction establishes the relation:

$$M_n = m_{2n-1}.$$

It is also possible to relate the number of open meanders of an even order to closed meanders. An open meander of order $2n$ can be completed to a closed meander of order n by joining the two ends of the curve at infinity and orienting the line appropriately, as illustrated in Figure 2.2. This operation is not bijective since in the reverse operation the closed curve can be broken along any segment incident to the infinite face in the upper half-plane. Each choice generates a different open meander.

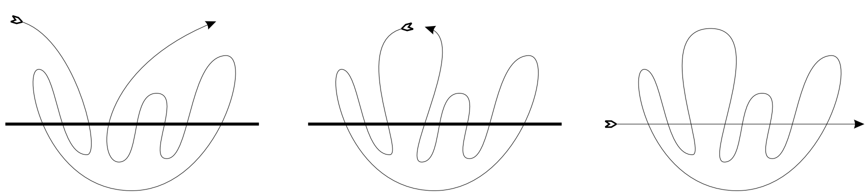


Figure 2.2: Closed meanders and even order open meanders

We thus have the relation that the number of open meanders of order $2n$ is the sum, over all closed meanders of order n, of the number of pieces of the closed curve in the upper half-plane that are incident to the infinite face.

2.2 Using Arch Configurations

Arch configurations provide a convenient medium for obtaining a canonical representation of closed meanders. In a closed meander, the line partitions the closed curve into two pieces, an upper piece and a lower piece, corresponding to the two sides of the line when the configuration is drawn with the line oriented from left to right. The pieces, each taken together with the line, are referred to as the upper and lower configurations of the meander. Each of these is homeomorphic to a unique arch configuration, the existence of which requires a topological argument beyond the scope of this essay. These are the canonical representations of the upper and lower configurations. The canonical representation of a closed meander is the unique meandric configuration that is homeomorphic to the original meander and in which both the upper and lower configurations are in canonical form.

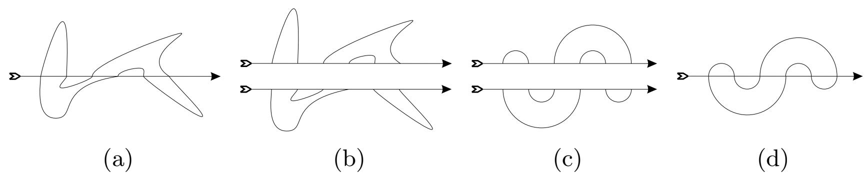


Figure 2.3: The canonical form of a closed meander.

Figure 2.3 illustrates this representation. A closed meander (a) is split into an upper configuration and a lower configuration given in (b). An arch configuration is constructed homeomorphic to each piece to obtain in (c) a canonical representation of the upper and lower configurations. Given two arch configurations of the same order, we can *superpose* them by drawing one on each side of the line and identifying their base points. Superposing the canonical representations produces the canonical representation of the original meander in (d).

To find the canonical representation of a closed meander of order n, consider the following construction. Homeomorphically deform the configuration to make the points of intersection between the curve and the line equally spaced along the line, while keeping the line fixed. Deleting the intersection points between the curve and the line partitions the closed curve into $2n$ connected components, n on each side of the line. Replace each of these components by a semicircular arch on the same side of the line, with the same points of intersection with the line. The upper and lower configurations of the resulting configuration are easily seen to both be arch configurations of order n. Showing that the final configuration is homeomorphic to the initial configuration requires repeated invocations of the Jordan Curve Theorem.

The construction provides an injective function from classes of closed meanders of order n to ordered pairs of arch configurations of order n, corresponding to the upper and lower arch configurations. We notice that not every such pair of arch configurations is the image of a closed meander: in particular, taking the same configuration for the upper configurations and the lower configurations yields n closed curves instead of a single closed curve. This motivates the following generalization of meanders.

Definition 2.2.1. *A* meandric system *is the superposition of an ordered pair of arch configurations of the same order, with the first configuration as the upper configuration, and the second configuration as the lower configuration.*

The order of a meandric system is the order its underlying arch configurations. To work with meandric systems, we use $M_n^{(k)}$ to denote the number of k component meandric systems of order n. This is a natural generalization of meandric numbers, in the sense that $M_n = M_n^{(1)}$. Values of $M_n^{(k)}$ for small n and k are listed in Table A.4.

A sequence of polynomials is used to summarize the meandric system numbers. The n-th *meandric polynomial*,

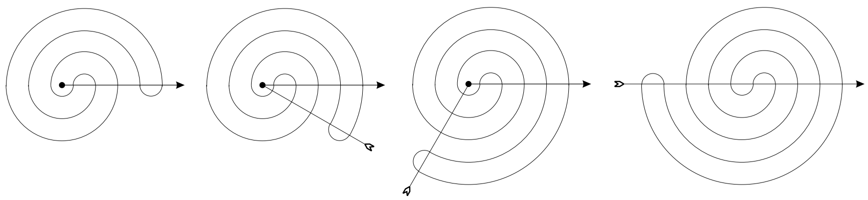
$$m_n(q) = \sum_{k=1}^{n} M_n^{(k)} q^k, \tag{2.1}$$

is the ordinary generating series of meandric systems of order n with respect to number of components. Given the polynomial $m_n(q)$, M_n can be recovered as the coefficient of q.

2.2.1 Embedding Semi-Meanders in Closed Meanders

Arch configurations can also be used to provide a canonical representation of semi-meanders. Consider the following construction illustrated in Figure 2.4.

Beginning with a semi-meander (a), create a second ray with the same origin as the first. The new ray can be created arbitrarily close to the original ray, such that it is distinct from the original ray but the semi-meandric configuration consisting of the closed curve and the original ray



(a) (b) (c) (d)

Figure 2.4: Semi-meanders specialize closed meanders

is equivalent to the configuration consisting of the closed curve and the new ray (b). By homeomorphically deforming the plane, the two rays can be separated further and aligned such that they are oriented in opposite directions (c). Interpreting this pair of rays as a line oriented in the same direction as the original ray, we have constructed a meandric configuration (d).

In this presentation, the configurations swept out between the two rays is seen to be a rainbow configuration. Moreover, beginning with a closed meander with a rainbow configuration for its lower arch configuration, it is possible to reverse the transformation to obtain a semi-meander by interpreting the line as a pair of rays originating from the center of the rainbow configuration and deleting one of the rays. This provides a natural embedding of semi-meanders in meanders and establishes the following proposition.

Proposition 2.2.2. *Meandric numbers and semi-meandric numbers satisfy the relation*

$$\overline{M}_n \leq M_n. \tag{2.2}$$

By representing meanders as pairs of arch configurations, we obtain a natural representation for semi-meanders in terms of arch configurations. Semi-meanders can be viewed as the specialization of meanders to those with the rainbow configuration as their lower arch configuration. In addition, the closed curve of a semi-meander of order n intersects the ray n times and intersects the pair of rays $2n$ times, so the closed meander obtained by this correspondence is also of order n. It is this relationship that motivated a definition of order that at first appears to be an inconsistent between the two classes.

This representation also preserves winding number as an identifiable parameter. When a semi-meander is presented as a closed meander with the rainbow configuration as its lower configuration, the winding number can be recovered as the number of arches passing over the midpoint of the upper arch configuration.

Parallel to the definition of a meandric system, we define a *semi-meandric system* as the superposition of an arbitrary upper arch configuration with a lower rainbow configuration.

2.2.2 Embedding Closed Meanders in Semi-Meanders

Closed meanders also have a natural embedding in semi-meanders, though in this embedding, the order is not preserved. A semi-meander of order $2n$ with winding number zero can be drawn so the ray can be extended to a line without intersecting the closed curve. A closed meander of order n is obtained by orienting the line consistently with the ray.

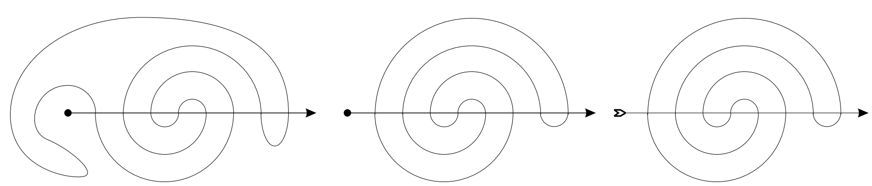


Figure 2.5: Closed meanders specialize semi-meanders

Figure 2.5 illustrates this interpretation: a semi-meander of order 8 with winding number zero (left) is drawn such that the ray can be extended to a line without intersecting the closed curve and the resulting configuration is interpreted as a closed meander of order 4 (right) by replacing the ray by the oriented line containing it.

In a similar manner, a closed meander can be interpreted as a semi-meander with winding number zero by picking any point on the line that precedes all intersections, and using it as the origin of the ray of a semi-meandric configuration using the same closed curve. In terms of diagrammatic representations, the construction is trivial, amounting to replacing the tail of an arrow, marking the direction of a line, by a dot, indicating the origin of a ray.

Using this interpretation, we can interpret closed meanders as a specialization of semi-meanders. Closed meanders are those semi-meanders with a winding number of zero. The following proposition is an immediate consequence.

Proposition 2.2.3. *Meandric numbers and semi-meandric numbers satisfy the relation*

$$M_n \leq \overline{M}_{2n}. \tag{2.3}$$

2.2.3 Bounding Meandric Numbers

We can obtain both an upper bound and a lower bound for meandric numbers in terms of arch configurations.

Lemma 2.2.4. *The numbers M_n and C_n satisfy the relation*

$$C_n \leq M_n \leq {C_n}^2.$$

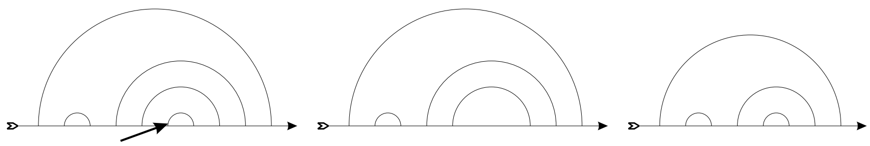
Proof. Meandric systems of order n are in bijective correspondence with ordered pairs of arch configurations of order n, of which there are ${C_n}^2$. So the inequality

$$M_n \leq {C_n}^2$$

is a consequence of meanders being a subset of meandric systems of the same order.

To establish the lower bound, and complete the proof, we show that every arch configuration occurs as the upper configuration of some meander. We begin with the observation that every arch configuration contains an arch such that no arch lies between its feet. Call such an arch a *minimal arch.*

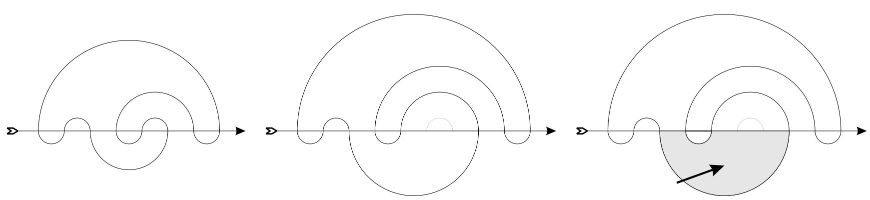
We employ a construction that uses one meander to generate several closed meanders of the next higher order. Consider two arch configurations such that the second is derived from the first by erasing a minimal arch and redrawing the resulting configuration so the feet of the arches are again equally spaced. Figure 2.6 illustrates this derivation: beginning with (a), (b) is obtained by deleting the minimal arch indicated with an arrow, redrawing (b) as (c) completes the derivation.



(a) (b) (c)

Figure 2.6: Minimal arch deletion

Starting with a meander of order $n-1$ with the derived arch configuration as its upper configuration, we can construct a meander of order n with the first arch configuration as its upper configuration. Figure 2.7 illustrates this construction. Beginning with an arbitrary closed meander with the derived arch configuration as its upper configuration (a), we obtain an equivalent meander, with the arches repositioned to accommodate *in situ* the missing arch (b), and locate the region of the lower configuration incident with the feet of the missing arch (c).



(a) (b) (c)

Figure 2.7: Deforming a meander to accomodate a missing arch

A meander of order n with the first arch configuration as its upper configuration is obtained by picking any lower arch incident to this region and stretching it to cross the line and replace the deleted arch. In general the choice is not unique. Figure 2.8 uses an arrow to indicate the two lower arches that can be selected, and illustrates the result of the construction in each case.

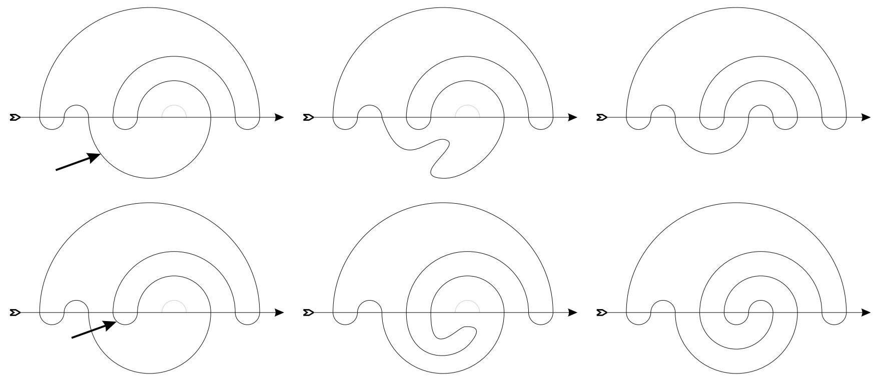


Figure 2.8: Stretching an arch across the line

This construction can be used to produce a closed meander with any specified arch configurations as its upper configuration. To do this, construct a sequence of arch configurations beginning with the specified configuration and ending with the configuration consisting of a single arch, such that each configuration is obtained from its immediate predecessor by deleting a minimal arch. Figure 2.9 illustrates such a sequence beginning with the arch configuration from the previous example.

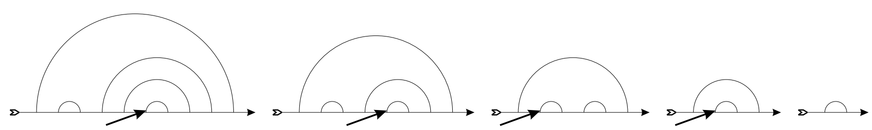


Figure 2.9: A sequence minimal arch deletions

The configuration with a single upper arch is uniquely completed by the configuration with a single lower arch. Beginning with this meander use the preceding construction to produce a meander for each arch configuration in the sequence. Such a sequence is illustrated in Figure 2.10 with an arrow indicating the arch that is stretched at each step.

Since this construction can be applied beginning with any arch configuration to obtain a meander with the specified configuration as its upper configuration, we conclude that, every arch configuration is the upper con-

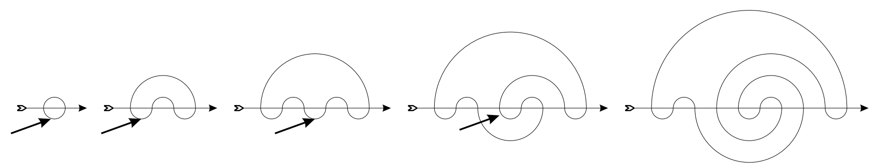


Figure 2.10: A sequence of meanders

figuration of at least one meander. This establishes the lower bound

$$C_n \leq M_n. \qquad \square$$

Arch configurations are easily shown to be in bijective correspondence with several other classes of combinatorial objects of known cardinality. For completeness we provide a direct derivation of the arch configuration numbers.

Lemma 2.2.5. *The number of arch configurations of order* n, C_n *is* $\frac{1}{n+1}\binom{2n}{n}$*, the* n*-th Catalan number.*

Proof. We consider the null configuration (with no arches) to be an arch configuration. Let $\mathcal{A}$ denote the class of arch configurations, and $\mathcal{B}$ denote the class of arch configurations with a single arch incident with the infinite face. An arbitrary arch configuration can be decomposed as a finite sequence of arch configurations of class $\mathcal{B}$. The decomposition is reversible since a sequence of arch configurations of class $\mathcal{B}$ can be concatenated to produce an arch configuration of general type. This establishes the bijection:

$$\mathcal{A} \xrightarrow{\sim} \dot{\bigcup_{i=0}^{\infty}} \mathcal{B}^i$$

which preserves the number of arches. Thus if $A(x)$ and $B(x) \in \mathbb{Q}[[x]]$, the ring of formal power series over $\mathbb{Q}$, are the ordinary generating series for $\mathcal{A}$

and $\mathcal{B}$, respectively, then

$$A(x) = \sum_{i=0}^{\infty} B(x)^i = \frac{1}{1 - B(x)}.$$

But an arch configuration of general type can be obtained from one of class $\mathcal{B}$ by deleting the arch incident with the infinite face. Since the process is reversible, given an arch configuration of general type we can obtain a unique arch configuration of class $\mathcal{B}$ by adding an arch spanning the entire configuration, we have the additional relation,

$$B(x) = xA(x),$$

and conclude that $A(x)$ satisfies the equation

$$A(x) = \frac{1}{1 - xA(x)},$$

or equivalently

$$A(x) - 1 = xA(x)^2.$$

Letting $D(x) = A(x) - 1$, we have

$$D(x) = x(1 + D(x))^2,$$

and by Lagrange's Implicit Function Theorem [8, Thm. 1.2.4],

$$\begin{aligned} D(x) &= \sum_{n \geq 1} \frac{x^n}{n} [\lambda^{n-1}](1+\lambda)^{2n} \\ &= \sum_{n \geq 1} \frac{1}{n} \binom{2n}{n-1} x^n \\ &= \sum_{n \geq 1} \frac{1}{n+1} \binom{2n}{n} x^n \end{aligned}$$

is the unique solution to this equation in $\mathbb{Q}[[x]]$. From this we see that

$$A(x) = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n,$$

and so

$$C_n = [x^n]A(x) = \frac{1}{n+1} \binom{2n}{n},$$

as required. $\square$

We have an immediate corollary.

Corollary 2.2.6. *The number of meandric systems of order n is* $\left(\frac{1}{n+1}\binom{2n}{n}\right)^2$.

We also have a rough picture of the general behaviour of the meandric numbers. From Stirling's approximation, the asymptotic behaviour of C_n is known to be $C_n \sim C\frac{4^n}{n^{3/2}}$ for a constant C. It is thus reasonable to assume that the asymptotic behaviour of M_n is

$$M_n \sim C\frac{R^n}{n^\alpha} \tag{2.4}$$

for some constants C, R, and α. Indeed, it is shown in [11] that there exists a constant A_M such that for all A with $0 < A < A_M$, $A^n < M_n \leq (A_M)^n$.

From (2.2) and (2.3), we can bound the semi-meandric numbers by $M_n \leq \overline{M}_n \leq M_{2n}$ and can conclude that they also exhibit exponential growth. It is conjectured that

$$\overline{M}_n \sim \overline{C}\frac{\overline{R}^n}{n^{\overline{\alpha}}} \tag{2.4$'$}$$

for some constants $\overline{C}$, $\overline{R}$, and $\overline{\alpha}$. For reasons stemming from interpretations of meanders and semi-meanders in statistical mechanics, it is further conjectured, in [5], that $R = \overline{R}^2$.

2.3 Filtering Meanders From Meandric Systems

By considering a meandric system as a pair of arch configurations of the same order we have determined that the number of such systems is ${C_n}^2$. We now give an expression, as described in [11], for the generating series for meandric systems with respect to twice their order,

$$B(x) = \sum_{n \geq 0} {C_n}^2 x^{2n}.$$

Proposition 2.3.1. *The generating series $B(x)$ has the expression,*

$$B(x) = \frac{1}{4x^2}\left(-1 + \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1 - 8x\cos\phi + 16x^2}\,\mathrm{d}\phi\right). \tag{2.5}$$

Proof. We work in $\mathbb{C}((x))$, the field of formal Laurent series over $\mathbb{C}$, and make use of a result due to Parseval, see for example [11].

Lemma (Parseval 1805). *If $f(x) = \sum_{n\geq 0} a_n x^n$ is analytic in a neighbourhood of zero in $\mathbb{C}$, then $F(x) = \sum_{n\geq 0}(a_n x^n)^2$ is analytic in a neighbourhood of zero and representable as*

$$F(x) = \mathrm{Res}_{\lambda=0}\,\frac{f(\lambda x) f(\lambda^{-1} x)}{\lambda}.$$

By considering the natural embedding of $\mathbb{Q}[[x]]$ in $\mathbb{C}((x))$, we can view $A(x)$, the generating series for arch configurations as the function,

$$A(x) = \sum_{n\geq 0} C_n x^n = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

which is analytic on the disc $D = \{x \in \mathbb{C} \colon |x| < \frac{1}{4}\}$. So, by the lemma, with

A and B taking the roles of f and F, respectively,

$$\begin{aligned}
B(x) &= \mathrm{Res}_{\lambda=0} \frac{A(\lambda x)A(\lambda^{-1}x)}{\lambda} \\
&= \mathrm{Res}_{\lambda=0} \frac{1}{\lambda} \frac{1-\sqrt{1-4\lambda x}}{2\lambda x} \frac{1-\sqrt{1-4\lambda^{-1}x}}{2\lambda^{-1}x} \\
&= \frac{1}{4x^2} \mathrm{Res}_{\lambda=0} \frac{1}{\lambda}(1-\sqrt{1-4\lambda x})(1-\sqrt{1-4\lambda^{-1}x}) \\
&= \frac{1}{4x^2} \left[\mathrm{Res}_{\lambda=0} \frac{1}{\lambda}(1-\sqrt{1-4\lambda x}) - \mathrm{Res}_{\lambda=0} \frac{1}{\lambda}(\sqrt{1-4\lambda^{-1}x}) \right. \\
&\qquad\qquad \left. + \mathrm{Res}_{\lambda=0} \frac{1}{\lambda}(\sqrt{1-4\lambda x - 4\lambda^{-1}x + 16x^2}) \right] \\
&= \frac{1}{4x^2} \left(-1 + \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - 8x\cos\phi + 16x^2}\, \mathrm{d}\phi \right)
\end{aligned}$$

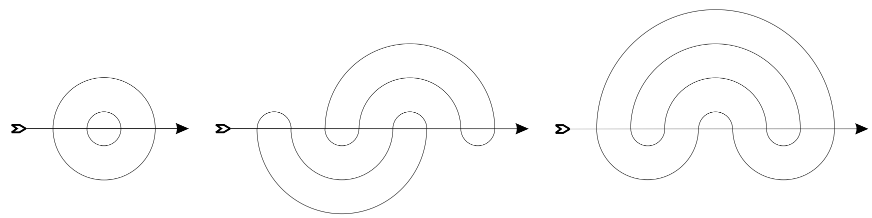
The first two residues are seen to be 0 and 1, respectively, while the final residue is obtained by integrating around the perimeter of the unit circle with λ replaced by $\cos\phi + i\sin\phi$. □

Given the relative ease with which we obtain a solution to the problem of enumerating systems of meanders, and that a system of meanders is a multi-component meander, one might hope to be able to exploit this knowledge to obtain an exact solution to the number of single component, or connected meanders. Indeed, under the appropriate combinatorial conditions, there is a simple relation between the generating series for connected objects and the generating series for the set of objects constructed from a finite collection of connected objects. We consider two classes of objects for which the number of connected objects can be obtained from the number of general objects, and show that meandric systems do not fall into either of these classes.

For the first class of objects, a general object is obtained from connected objects by Cartesian product, and can be decomposed into a sequence of connected objects. For this class, the ordinary generating series for general objects $G(x)$ and the ordinary generating series for connected objects $C(x)$ satisfy the relation

$$G(x) = \frac{1}{1-C(x)}. \tag{2.6}$$

Meanders and meandric systems are not in this class of objects. Any sequence of meanders can be concatenated along the line to form a meandric system, but not every meandric system takes this form. Two closed curves may be nested as in Figure 2.11 (a), intertwined as in Figure 2.11 (b), or both as in Figure 2.11 (c), for example.



(a) (b) (c)

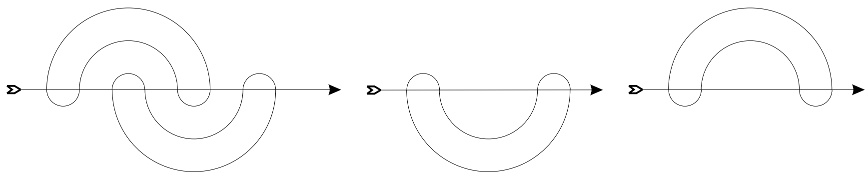
Figure 2.11: Meandric systems that are not formed by concatenation

For the second class of objects, a general object is obtained from connected objects by $\circledast$-product [8, Def. 3.2.9], and can be decomposed into a collection of connected objects on disjoint sets of labels. In this case, $G(x)$, the exponential generating series for the general objects, and $C(x)$, the exponential generating series for the connected objects, satisfy the relation

$$G(x) = \exp(C(x)). \tag{2.7}$$

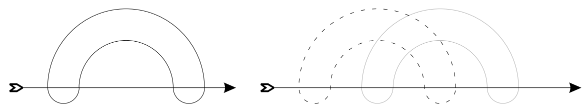
Considering the base points as the labelled objects, every component of a meandric system is a closed meander on a subset of the labels: the meandric system Figure 2.12 (a) is the superposition of the meander Figure 2.12 (b) on the second and forth pairs of intersections, with the meander Figure 2.12 (c) on the first and third pairs of intersections, for example. Meandric systems do not, however, fall into this second class of objects.

The superposition of two copies of the meander in Figure 2.13 (a) using the same partition as in Figure 2.12 (a) produces a configuration, Figure 2.13 (b), that is not a meandric system; the upper configuration is not planar. The allowable configurations for one component are dependent on the form of the other components, and this violates the conditions under which (2.7) holds.



(a) (b) (c)

Figure 2.12: A meandric system is a collection of meanders



(a) (b)

Figure 2.13: A collection of meanders that is not a meandric system

There does not appear to be an elementary combinatorial description of the operation of constructing a meandric system from meanders. As a result, there is no known way to obtain the meandric numbers for meandric system numbers. In particular, the generating series for meanders cannot be related to the generating series for meandric systems by either (2.6) or (2.7).

2.4 Automorphisms of Meanders

Automorphisms on the class of meanders can be used constructively in algorithms for constructing meanders of a given order. They can also be used to simplify an exhaustive enumeration of meanders by eliminating redundancy. Given an automorphism on the class of meanders, it is possible to encode the entire class by a list that includes only a single element of each orbit under the action of the automorphism. If the automorphism is easily computable, then analysis on the entire class can be carried out by iterating through the list of orbit representatives and expanding each orbit in turn. For certain automorphisms, it is possible to derive congruences satisfied by the meandric numbers by considering the size of orbits under the action of the automorphism.

Two classes of automorphisms are particularly useful: those defined by rigid transformations of the plane, and those defined by cyclic shifts of the intersection points. These are now considered in greater detail.

2.4.1 Rigid Transformations

The most obvious class of automorphisms consists of rigid transformations of the plane. In particular, the closed curve of a meander can be reflected in ($\updownarrow$) or along the line ($\leftrightarrow$), or reflected through a point ($\circlearrowright = \updownarrow \circ \leftrightarrow = \leftrightarrow \circ \updownarrow$) i.e. rotated through 180°, to produce a new meander of the same order. These three transformations are illustrated in Figure 2.14 (b), (c), and (d), respectively, for the meander given in Figure 2.14 (a).

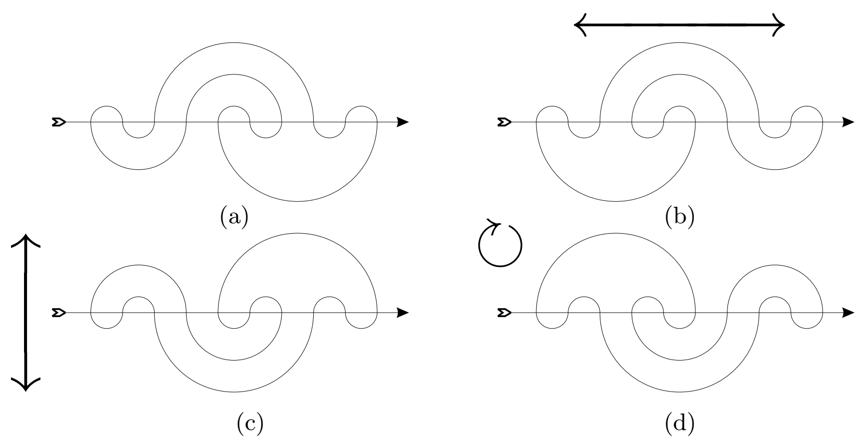


Figure 2.14: Rigid automorphisms of meanders

The automorphism $\updownarrow$ has an especially convenient realization in terms of arch configurations, since its action corresponds to interchanging the upper and lower configurations. We establish some properties of this automorphism.

Proposition 2.4.1. *The meander of order one is the only closed meander that is invariant under the action of* $\updownarrow$.

Proof. For a meander to be invariant under the action of $\updownarrow$, every arch in its upper configuration has a counterpart in the lower configuration that sits on the same base points. The superposition of any arch from the upper configuration with its counterpart from the lower configuration forms a closed curve. Since the meander has only a single closed curve, it must consist of just this single pair of arches. $\square$

Corollary 2.4.2. *For $n \geq 2$, the meandric numbers satisfy the congruence*

$$M_n \equiv 0 \pmod 2.$$

Proof. Since $\updownarrow$ has order two, every orbit, except the one consisting of the meander of order one, is of order two. So a list, consisting of one representative of each orbit, contains exactly half of the meanders of every order greater than one. This establishes the congruence. $\square$

To construct a list of orbits under the action of $\updownarrow$, we need only have a convenient means for identifying a canonical representative of each orbit. This can be accomplished, for example, by taking as representatives those closed meanders for which the leftmost arch in the upper configuration is longer than the leftmost arch in the lower configuration. Figure 2.15 gives a complete list of representatives of the meanders of order 3 under this choice. Contrast this to the list of meanders of order 3 given in Figure 1.7.

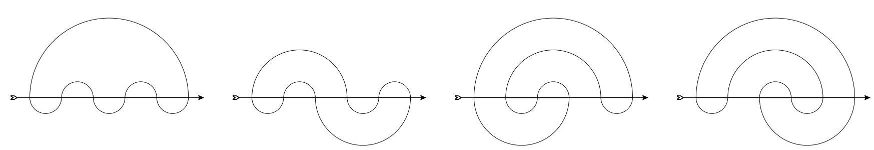


Figure 2.15: Representative meanders of order 3

The other two automorphisms do not act as conveniently. In particular, they have fixed points. The first two meanders in Figure 2.16 are fixed by $\leftrightarrow$, while the third is fixed by $\circlearrowright$. As a result, limiting the list to one representative of each orbit under either of these automorphisms does not reduce the size of the list by a full factor of two, and analyzing these automorphisms does not produce any new congruences.

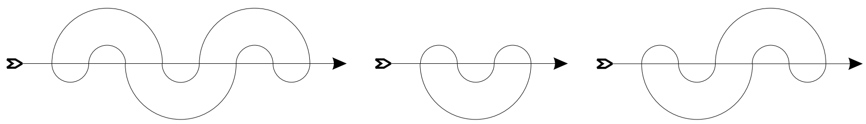


Figure 2.16: Fixed points of rigid automorphisms of the plane

The additional space savings afforded by considering only a single representative of each orbit under the action of the group consisting of these three automorphisms together with the identity, can still be significant in carrying out an exhaustive enumeration. These orbits form an identifiable class of objects in their own right and correspond to a class of meanders where the oriented line is replaced by an unoriented line in the definition, and equivalence is taken up to homeomorphism with the plane considered as a subset of three-dimensional Euclidean space.

2.4.2 Cyclic Shifts

A second useful class of automorphisms is the cyclic shift of the base points along with a canonical adjustment to the meander. A left cyclic shift is performed by breaking the closed curve at its leftmost intersection with the line and rejoining the free ends to form the rightmost intersection. The inverse of this operation, the right cyclic shift, clearly exists, so the cyclic shift is an automorphism. Figure 2.17 illustrates the stages of a left cyclic shift applied to the leftmost diagram.

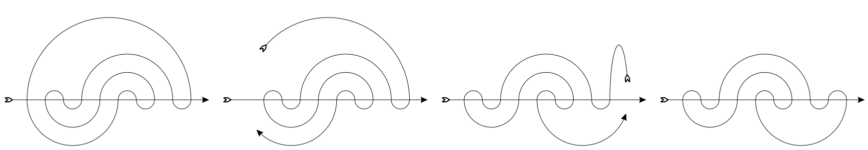


Figure 2.17: A cyclic shift

Applying $2n$ shifts to a closed meander of order n restores the meander to its original configuration, so the order of every orbit divides $2n$. A complete orbit under the action of the shift operation is shown in Figure 2.18. As with rigid transformations, not every meander is in an orbit of maximum

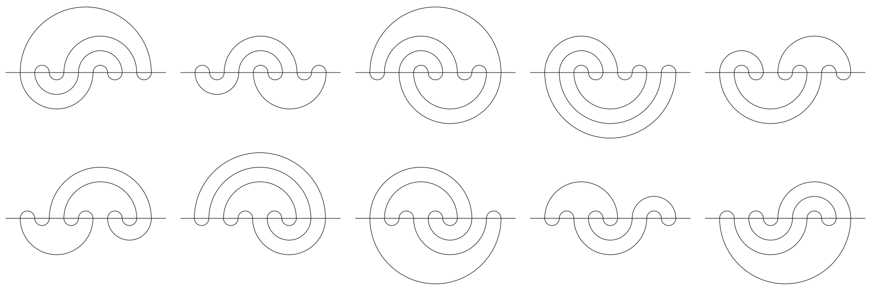


Figure 2.18: A complete orbit under the action of cyclic shift

order, although the meander of order one is again the only meander that is fixed by the automorphism.

Proposition 2.4.3. *For every order greater than one, there is a single orbit of order two and no orbit of order one under the action of cyclic shift.*

Proof. Consider a meander M of order n. If the base points of M are labelled $\{1, 2, \ldots, 2n\}$ according to the orientation of the line, then the upper arch configuration of M has at least one minimal arch. Without loss of generality, its base points are $i-1$ and i.

Let $\Omega(M)$ denote the image of M under the action of a right cyclic shift. So $\Omega(M)$ has an upper arch with base points i and $i+1$. Thus, if M is fixed by Ω, then $i-1 \equiv i+1 \pmod{2n}$ and so $n=1$.

Similarly, if M is fixed by Ω^2 then its upper and lower configurations must be fixed by the induced action of Ω^2. But the only arch configurations fixed by Ω^2 are the configuration consisting entirely of minimal arches and the configuration obtained by applying a cyclic shift to it. The orbit of

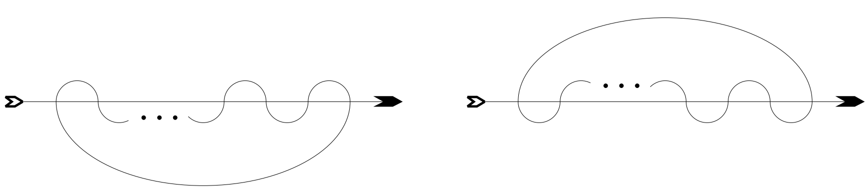


Figure 2.19: The orbit of order two under the action of cyclic shift

order two consists of the two meanders with one of these configurations as an upper configuration and the other as a lower configuration. It is given in Figure 2.19. ☐

Corollary 2.4.4. *For p an odd prime,*

$$M_{p^k} \equiv 2 \pmod{2p}.$$

Proof. We consider the orbits of meanders of order p^k under the action of cyclic shift. Every orbit has order dividing $2p^k$, so p divides the order of every orbit, other that the one of length 2. Thus

$$M_{p^k} \equiv 2 \pmod{p}.$$

Combining this with Corollary 2.4.2, we conclude that,

$$M_{p^k} \equiv 2 \pmod{2p}.$$ ☐

As with rigid transformations, the orbits under the action of cyclic shift have a natural interpretation as a class of objects in their own right. Since only the cyclic order of the intersections between the curve and the line is relevant in identifying an orbit, an orbit can be represented by replacing the line in the definition of a meander with an oriented closed curve. Figure 2.20 illustrates this encoding. This class of objects should be considered as embedded on a sphere, since a cyclic shift would otherwise involve passing a segment of the outer curve through the point at infinity.

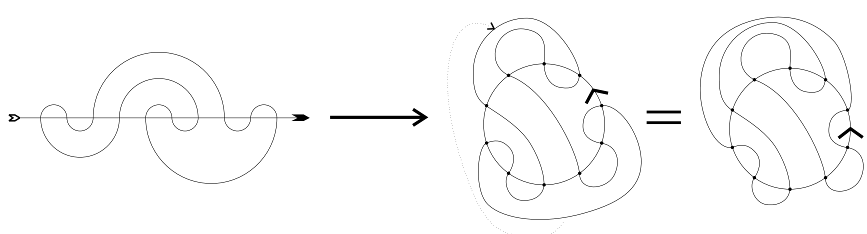


Figure 2.20: Representing an orbit under cyclic shift

By replacing the oriented line by a second closed curve, and replacing the plane by a sphere, we obtain a parallel to open meanders, in that the two curves are equivalent in the definition. In fact, the orbit of a closed meander under cyclic shift corresponds to the configuration obtained by taking the one point compactification of the open meander of odd order found under the equivalence in Section 2.1.

2.5 Enumeration by Tree Traversal

Since, to date, there is no efficient way to compute meandric or semi-meandric numbers exactly, one approach for obtaining asymptotic results has been to use the conjectured asymptotic form $M_n \sim C\frac{R^n}{n^\alpha}$ and experimentally determine values for the constants C, R, and α. This approach requires the knowledge of a large number of terms. The most effective way to obtain these numbers has been to list all the meanders of the required form and count them.

One method for enumerating a class of objects is to construct a forest, such that the vertices at depth n are exactly the objects of order n. We consider only the case where there is a single object of order one, and the forest is a tree. In order to construct such a tree, we need only have a method for identifying incidence. This can be accomplished by defining a function mapping objects of order $n > 1$ to objects of order $n - 1$. By means of such a function, a tree can be formed by taking the unique object of order one as the root, and defining the children of a given object to be its preimages under the function.

If the preimages are easily computable, then it is possible to enumerate all objects of a given order by using standard tree traversing algorithms. This traversal can be done with an amount of memory polynomial in the desired order, and in time proportional to the number of objects of that order.

2.5.1 A Tree of Semi-Meanders

To construct such a function for the class of semi-meanders, we consider semi-meanders as closed meanders with a lower rainbow configuration. In this presentation, define the image of a semi-meander to be the semi-meander obtained by breaking the outer arch of the lower configuration, contracting the two free ends across the line, and rejoining them to create a new arch in the upper configuration. This is illustrated in Figure 2.21.

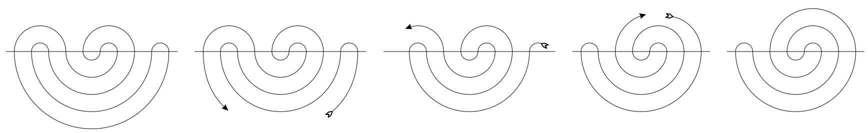


Figure 2.21: A function on semi-meanders

This function satisfies the condition of having an easily computable preimage. A semi-meander has one preimage for each arch of the upper configuration that is incident with the infinite face. For a given arch, the corresponding preimage is obtained by breaking the arch, stretching the two free ends, one around each end of the configuration, and rejoining them to create a new outer arch in the lower configuration. Since every semi-meander of order ≥ 2 has at least 2 upper arches incident to the infinite face, we have the immediate consequence that, for $n \geq 2$,

$$\overline{M}_n \leq 2\overline{M}_{n+1}.$$

Figure 2.22 illustrates the tree of semi-meanders obtained using this function. Notice that the left half of this tree is the mirror image of the right half. Mirroring a semi-meander in this presentation corresponds to mirroring a semi-meander in the standard presentation across the line. Using this tree, it is thus possible, by mirroring every semi-meander, to enumerate all semi-meanders while traversing only the branch corresponding one of the semi-meanders of order 3.

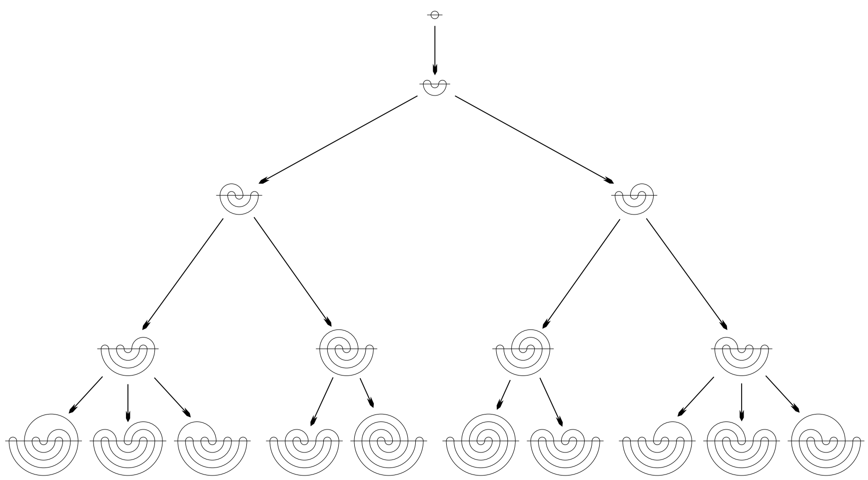


Figure 2.22: The tree of semi-meanders of order ≤ 5

This method can be extended to enumerate all systems of semi-meanders. In this case, each semi-meandric system has, as its children, all of its preimages under the function and the system obtained by adding a circle surrounding the original system.

2.5.2 A Tree of Meanders

A similar construction can be used to generate all closed meanders. To define the incidence function, we use the observation that every closed meander has at least one minimal arch in its upper configuration.

To find the image of a closed meander, apply cyclic shifts, if necessary, until a minimal arch appears as the leftmost arch of the upper configuration. For all four meanders in Figure 2.23, the rightmost meander is obtained by this step. The image of the original meander is obtained by pushing this minimal arch across the line. Figure 2.24 illustrates this final step.

As with semi-meanders, the preimage is easily computed. A meander has one primary child per arch of the lower configuration incident with the infinite face. For a given arch, the corresponding child is found by pulling the arch across the line to the left of the existing configuration. The remaining

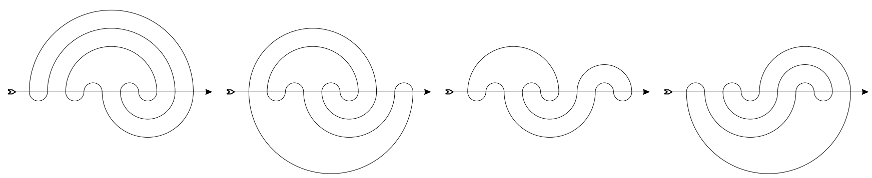


Figure 2.23: Shifting a meander to obtain a minimal arch

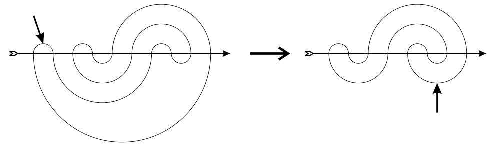


Figure 2.24: The image of a meander

children are found by applying reverse cyclic shifts to the primary children until the upper configuration has a single arch incident with the infinite face, a subsequent shift would result in a different minimal arch. Since every meander has at least one child, we conclude that the sequence of meandric numbers is monotonically increasing.

Chapter 3

The Symmetric Group

Meanders have a natural representation as permutations. This chapter uses this representation to present a combinatorial construction for meanders. The meandric and semi-meandric numbers can then be expressed in terms of characters of the symmetric group. Throughout this chapter, elements of $\mathfrak{S}_n$ are respresented by their disjoint cycles representation.

3.1 Representing Meanders As Permutations

Given a closed meander of order n, we represent it such that the line is horizontal and oriented from left to right, the closed curve is oriented such that at the first intersection between the curves it is directed from bottom to top. The intersections between the two curves have a natural labelling by the integers $\{1, 2, \ldots, 2n\}$, defined by their ordering along the line. These labels also have a cyclic order defined by their order along the closed curve. This cyclic order, when interpreted as a a full cycle in $\mathfrak{S}_{2n}$, is referred to as a *meandric permutation*. There is a one-to-one correspondence between meanders and meandric permutations. Figure 3.1 shows the meandric permutation (1 10 9 4 3 2 5 8 7 6).

Interest in meandric permutations predates modern interest in the enumerative theory of meanders. They were discussed for instance by Rosen-

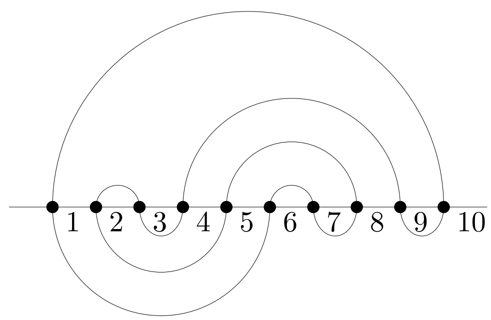


Figure 3.1: The meandric permutation (1 10 9 4 3 2 5 8 7 6)

stiehl in [12] as *planar permutations.* These permutations occur in the analysis of geographical data, and have the property that they can be sorted in linear time [12] . Enumerative information about meanders can be used to bound the performance of algorithms dealing with this sorting.

It is an immediate consequence of the definition, that every meandric permutation is in the conjugacy class $\mathcal{C}_{(2n)}$ of $\mathfrak{S}_{2n}$, where n is the order of the corresponding closed meander. A natural question is how to determine whether a given permutation in $\mathfrak{S}_{2n}$ of cycle type $(2n)$ is a meandric permutation. We have the following necessary condition.

Proposition 3.1.1. *If $\pi \in \mathfrak{S}_{2n}$ is a meandric permutation, then π^2 is a permutation of cycle type $[n^2]$, where one cycle is on the odd symbols and the other is on the even symbols.*

Proof. We consider the upper and lower arch configurations of the meander corresponding to π. Every arch spans an integral number of smaller arches, each of which has an even number of base points. Thus every arch has one foot on an even symbol and the other on an odd symbol.

Following the closed curve involves steps that alternate between upper and lower arches, so as a consequence, every meandric permutations alternates between even and odd symbols. The result follows. $\square$

This condition does not characterize meandric permutations. Consider the permutation (1 4 3 6 5 2), with square (1 3 5)(2 4 6) which is not a meandric permutation. Attempting to draw the meander corresponding to (1 4 3 6 5 2), one finds that the closed curve must be self intersecting in the upper half plane.

The difficulty in identifying meandric permutations is a consequence of the fact that planarity is a property of the upper and lower configurations rather than the meander as a whole. It is possible to recover the upper and lower configurations from a meandric permutation, as will be described later in this chapter, but there is no apparent way to do so algebraically.

3.1.1 Automorphisms of Meandric Permutations

The automorphisms described in Section 2.4 all have natural interpretations in terms of meandric permutations.

Reflecting a meander in the line reverses the orientation of the closed curve. The corresponding meandric permutation is thus obtained by reading the meandric permutation of the original meander in the reverse order. This is simply the inverse of the permutation obtained from the original meander. We conclude that the inverse of a meandric permutation is a meandric permutation.

To interpret reflection along the line and cyclic shifts in terms of permutations, we introduce two permutations,

$$\begin{aligned} &\tau = \tau_n = (1\ 2n)(2\ 2n{-}1)\cdots(n\ n{+}1) \qquad \text{and} \\ &\sigma = \sigma_n = (1\ 2\ \ldots\ 2n), \end{aligned} \tag{3.1}$$

corresponding respectively to the actions of reflection along the line and right cyclic shift on the labels.

Reflection along the line reverses the order of the labels and reverses the orientation of the closed curve. Similarly, a cyclic shift, shifts the labels and reverses the orientation of the curve. So, for π a meandric permutation, the images of π under reflection along the line and right cyclic shift are, respectively, $\tau^{-1}\pi^{-1}\tau$ and $\sigma^{-1}\pi^{-1}\sigma$. Since rotation of the closed curve is the composition of reflection across and along the line, the image of a rotation is $\tau^{-1}\pi\tau$. The following proposition is a consequence of reflection and cyclic shift being automorphisms.

Proposition 3.1.2. *If $\pi \in \mathfrak{S}_{2n}$ is a meandric permutation, then so are π^{-1}, $\tau^{-1}\pi\tau$, and $\sigma^{-1}\pi\sigma$; the class of meandric permutations is closed under inverse, conjugation by σ, and conjugation by τ.*

That the automorphisms from Section 2.4 are easily accessible in this encoding, is a testament to how natural the encoding is. It also suggests that the encoding is unlikely to make any additional structure more accessible.

3.2 Arch Configurations as Permutations

There is a second natural encoding of meanders in terms of the symmetric group. In order to deal with planarity, we use this alternate encoding. Instead of directly encoding meanders as permutations, we encode each meander as an ordered pair of arch configurations. Each arch configuration is then represented as a permutation. Using this encoding, it is possible to describe compactly which pairs of permutations correspond to meanders.

The permutation representation of arch configurations is a natural parallel to the representation of meanders as permutations. Each arch is a transposition on its endpoints, and the entire configuration is the product of these disjoint transpositions. An arch configuration of order n is the product of n disjoint transpositions in $\mathfrak{S}_{2n}$. As with meanders, the base points of the arches are labelled according to the orientation of the line. Figure 3.2 gives the arch configuration corresponding to the permutation $(1\ 10)(2\ 3)(4\ 9)(5\ 8)(6\ 7)$.

There are two steps in identifying which pairs of permutations in the conjugacy class $\mathcal{C}_{(2^n)}$ of $\mathfrak{S}_{2n}$ correspond to closed meanders. We must first identify which elements of $\mathcal{C}_{(2^n)}$ correspond to arch configurations. Then, given two such elements, we must determine whether the meandric system obtained by interpreting one as an upper arch configuration and the other

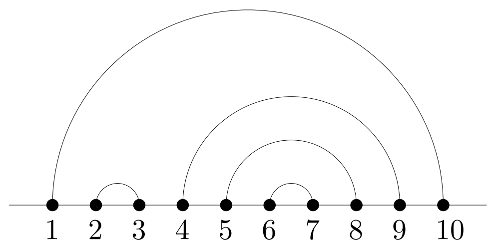


Figure 3.2: The arch configuration (1 10)(2 3)(4 9)(5 8)(6 7)

as a lower arch configuration is connected. These steps can be carried out algebraically. We deal with each step in turn.

3.2.1 Elements of $\mathcal{C}_{(2^n)}$ That Are Arch Configurations

Identifying arch configurations among elements of $\mathcal{C}_{(2^n)}$ is accomplished by interpreting them permutations as graphs, a class of objects for which planarity is more easily described. A graph corresponding to $\mu \in \mathcal{C}_{(2^n)}$ is constructed on $2n$ vertices labelled $\{1, 2, \ldots, 2n\}$ by creating an edge between every pair of vertices adjacent in the cyclic order $(1\ 2\ \ldots\ 2n)$, and adding an edge between every pair of vertices whose labels are in the same transposition of μ.

Such a graph is seen to have $3n$ edges, n corresponding to transpositions, and $2n$ corresponding to consecutive pairs of vertices. For the permutation to be an arch configuration, its graph must be planar. Since the graph is planar and has $2n$ vertices it has $n+2$ faces, by Euler's theorem. Figure 3.3 shows the graph of the arch configuration from Figure 3.2 together with its face boundaries.

We consider the cycles that bound the faces of the graph of an arch configuration. A single face, corresponding to the lower half-plane, is bounded entirely by edges defined by the cyclic order of the labels, while the boundary cycles of the remaining $n+1$ faces are seen to alternate between edges defined by transpositions and edges defined by the cyclic order of the base

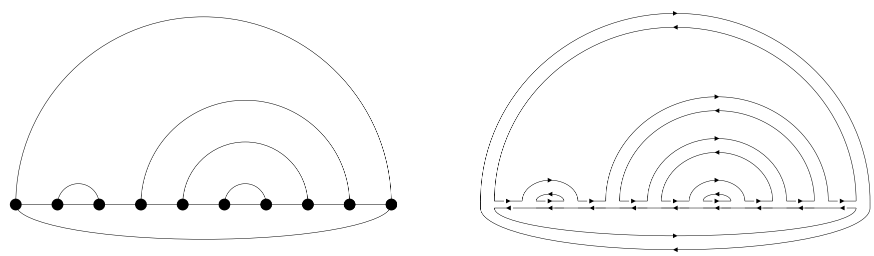


Figure 3.3: The graph of $\mu = (1\ 10)(2\ 3)(4\ 9)(5\ 8)(6\ 7)$ with its face boundaries

points. These are the cycles of $\sigma\mu$. Continuing the above example, $\sigma\mu = (1\ 3\ 9)(2)(4\ 8)(5\ 7)(6)(10)$ which consists of 6 cycles.

Letting $\kappa(\pi)$ denote the number of cycles in the disjoint cycle representation of the permutations π, we have the following characterization of the permutations that correspond to arch configurations.

Lemma 3.2.1. *The permutations in $\mathfrak{S}_{2n}$ that correspond to arch configurations form the set*

$$\{\mu \in \mathcal{C}_{(2^n)} :\ \kappa(\sigma\mu) = n+1\}.$$

Proof. If $\mu \in \mathcal{C}_{(2^n)}$ corresponds to an arch configuration, then by the above observation $\sigma\mu$ consists of $n+1$ cycles. Conversely, if μ does not correspond to an arch configuration, then at least two arches cross and $\sigma\mu$ consists of at most n cycles. $\square$

As expected, this class of permutations is closed under conjugation by τ and σ, operations that correspond respectively to reflection along the line and cyclic shift of the base points. We also note, that for $\sigma\mu$ to be a permutation with $n+1$ cycles on $2n$ elements, at least two cycles have length 1. These cycles correspond to the minimal arches of the arch configuration.

3.2.2 Completing the Characterization

Pairs of arch configurations of order n that form meandric systems with a single connected component can also be characterized algebraically. There is a single closed curve if and only if it is possible to walk through all the labels using a closed path alternating between arches from the upper configuration and arches from the lower configuration. Such a walk is a meandric permutation, so by taking it two steps at a time, by Proposition 3.1.1, we obtain two cycles of length n. We obtain the following characterization of which meandric systems are meanders.

Lemma 3.2.2. *If (μ_1, μ_2) is an ordered pair of transposition representations of arch configurations of order n, then the meandric system for which μ_1 represents the upper configuration and μ_2 represents the lower configuration is a meander if and only if $\mu_1\mu_2 \in \mathcal{C}_{(n^2)}$.*

Proof. The cycles of $\mu_1\mu_2$ correspond to walks through the labels taken two steps at a time along paths that alternate between arches described by transpositions of μ_1 and arches described by transpositions of μ_2. So if (μ_1, μ_2) corresponds to a meander, then by the above observation each cycle of this type is of length n, and $\mu_1\mu_2 \in \mathcal{C}_{(n^2)}$.

Conversely, if μ_1 and μ_2 are both transposition representations of arch configurations of order n, then every transposition of each contains an odd label and an even label. So if $\mu_1\mu_2 \in \mathcal{C}_{(n^2)}$, then each cycle consists only of labels with the same parity. A walk through all the labels can thus be formed by interleaving the two cycles and so (μ_1, μ_2) corresponds to a meander. $\square$

Noting that (μ_1, μ_2) represents a meander only if $\mu_1\mu_2$ is the square of the corresponding meandric permutation, we can see how to extract the upper and lower configurations from a meandric permutation. Consecutive pairs of symbols in the meandric permutation define the arches of the upper and lower configurations. The upper arch configuration is read by taking pairs of symbols starting with 1, while the lower configuration is read by taking symbols starting at either of the labels adjacent to 1.

Example 3.2.3. *For the meandric permutation $\pi = (1\ 10\ 9\ 4\ 3\ 2\ 5\ 8\ 7\ 6)$ of order 5, the upper configuration is $\mu_1 = (1\ 10)(9\ 4)(3\ 2)(5\ 8)(7\ 6)$, while the* *lower configuration is* $\mu_2 = (10\ 9)(4\ 3)(2\ 5)(8\ 7)(6\ 1)$. *Verifying the properties,* $\sigma\mu_1 = (1\ 3\ 9)(2)(4\ 8)(5\ 7)(6)(10)$ *and* $\sigma\mu_2 = (1\ 5)(2\ 4)(3)(6\ 8\ 10)(7)(8)$. *So* $\kappa(\sigma\mu_1) = \kappa(\sigma\mu_2) = 6$, *and* $\pi^2 = \mu_1\mu_2 = (1\ 9\ 3\ 5\ 7)(2\ 8\ 6\ 10\ 4) \in \mathcal{C}_{(n^2)}$.

We can now complete our combinatorial characterization of meanders with the following immediate consequence of Lemma 3.2.1 and Lemma 3.2.2.

Corollary 3.2.4. *The class of meanders of order n is in bijective correspondence with the set*

$$\{(\mu_1, \mu_2) \in \mathcal{C}_{(2^n)} \times \mathcal{C}_{(2^n)} \colon \ \kappa(\sigma\mu_1) = \kappa(\sigma\mu_2) = n+1,\ \mu_1\mu_2 \in \mathcal{C}_{(n^2)}\}.$$

Taking semi-meanders as the restriction of meanders to those with the rainbow configuration as a lower configuration, and noting that the rainbow configuration has the permutation representation τ, we obtain Corollary 3.2.5 as a specialization of Corollary 3.2.4 and achieve an algebraic characterization of semi-meanders.

Corollary 3.2.5. *The class of semi-meanders of order n is in bijective correspondence with the set*

$$\{\mu \in \mathcal{C}_{(2^n)} \colon \ \kappa(\sigma\mu) = n+1,\ \mu\tau \in \mathcal{C}_{(n^2)}\}.$$

The bijections described in Corollary 3.2.4 and Corollary 3.2.5 give us combinatorial expressions for meandric and semi-meandric numbers. Letting $\mathcal{C}^{(n+1)}$denote the set of all conjugacy classes of $\mathfrak{S}_{2n}$ with $n+1$ cycles, we obtain the expressions:

$$M_n = \sum_{\lambda_1,\lambda_2 \in \mathcal{C}^{(n+1)}} \sum_{\mu_1,\mu_2 \in \mathcal{C}_{(2^n)}} \delta_{[\sigma\mu_1],\lambda_1} \delta_{[\sigma\mu_2],\lambda_2} \delta_{[\mu_1\mu_2],(n^2)} \tag{3.2}$$

$$\overline{M}_n = \sum_{\lambda \in \mathcal{C}^{(n+1)}} \sum_{\mu \in \mathcal{C}_{(2^n)}} \delta_{[\sigma\mu],\lambda} \delta_{[\mu\tau],(n^2)}, \tag{3.3}$$

where $\delta_{a,b}$ is one if a and b are the same conjugacy class and zero otherwise. In this form, it looks as though it might be possible to determine the meandric numbers by working entirely within the center of the symmetric group, since, as a function of σ, the expression for M_n is constant on the conjugacy class (2^n). The same is not possible for the semi-meandric numbers, since the expression for $\overline{M}_n$ involves specific elements of two conjugacy classes.

3.3 Expression in Terms of Characters

For the remainder of the chapter, we will assume a familiarity with character theory. The reader is referred to [2] for a treatment of this subject. Characters of the symmetric group are class functions, and can be used to obtain an alternative expression for (3.2) and (3.3) that avoids the use of the δ-function.

The construction of characters involves representation theory and is of little interest to the present discussion. For our purposes, it suffices to note that the irreducible characters of a finite group G are functions $\chi^{(i)} \colon G \to \mathbb{C}$. They are class functions, that is they are constant on conjugacy classes of G, and are naturally indexed by the conjugacy classes of the group. In addition, for g in G and χ a character, $\chi(g) = \overline{\chi(g^{-1})}$.

We introduce some notation for a finite group G with k conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ of sizes $h^{(i)} = |C_i|$, $i = 1, 2, \ldots, k$. We let $\chi^{(i)}$ be the irreducible character indexed by conjugacy class $\mathcal{C}_i$ and let $\chi_j^{(i)}$ denote the evaluation of $\chi^{(i)}$ at any $g \in \mathcal{C}_j$. The irreducible characters (properly characters of irreducible representations) satisfy two orthogonality relations:

$$(1) \qquad \frac{1}{|G|} \sum_{i=1}^{k} h^{(i)}\, \chi_i^{(p)}\, \overline{\chi_i^{(q)}} = \delta_{p,q} \qquad \text{for } 1 \leq p, q \leq k$$

$$(2) \qquad \sum_{i=1}^{k} \chi_p^{(i)}\, \overline{\chi_q^{(i)}} = \frac{|G|}{h^{(p)}} \cdot \delta_{p,q} \qquad \text{for } 1 \leq p, q \leq k.$$

We use the second orthogonality relation for the group $G = \mathfrak{S}_{2n}$ to obtain an expression for meandric numbers and semi-meandric numbers in terms of characters of irreducible representations of the symmetric group.

Theorem 3.3.1. *If $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(k)}$ are the characters of the irreducible representations of the group $\mathfrak{S}_{2n}$, and σ and τ are as in* (3.1)*, then*

$$\begin{aligned} M_n = \sum_{\lambda_1,\lambda_2 \in \mathcal{C}^{(n+1)}} & \frac{|(2^n)|^2 \cdot |\lambda_1| \cdot |\lambda_2| \cdot |(n^2)|}{((2n)!)^5} \sum_{\mu_1,\mu_2 \in \mathfrak{S}_{2n}} \sum_{f,g,h,i,j=1}^{k} \\ & \chi^{(f)}(\mu_1)\chi^{(f)}(2^n)\chi^{(g)}(\mu_2)\chi^{(g)}(2^n)\chi^{(h)}(\sigma\mu_1)\chi^{(h)}(\lambda_1) \\ & \times \chi^{(i)}(\sigma\mu_2)\chi^{(i)}(\lambda_2)\chi^{(j)}(\mu_1\mu_2)\chi^{(j)}(n^2) \\ \overline{M}_n = \sum_{\lambda \in \mathcal{C}^{(n+1)}} & \frac{|(2^n)| \cdot |\lambda| \cdot |(n^2)|}{((2n)!)^3} \sum_{\mu \in \mathfrak{S}_{2n}} \sum_{h,i,j=1}^{k} \\ & \chi^{(h)}(\mu)\chi^{(h)}(2^n)\chi^{(i)}(\sigma\mu)\chi^{(i)}(\lambda)\chi^{(j)}(\mu\tau)\chi^{(j)}(n^2) \end{aligned}$$

Proof. For $\mathfrak{S}_{2n}$ the conjugacy classes and characters are naturally indexed by partitions of $2n$. For $v \vdash n$, the size of the conjugacy class $(1^{v_1}2^{v2}\cdots)$ with v_i cycles of length i for each i is

$$|(1^{v_1}2^{v2}\cdots)| = \frac{(2n)!}{\prod_i i^{v_i} v_i!}.$$

In addition, for every g in $\mathfrak{S}_{2n}$, g and g^{-1} are in the same conjugacy class. Thus for any character χ,

$$\overline{\chi(g)} = \chi(g^{-1}) = \chi(g)$$

so χ is real valued and in the case of $\mathfrak{S}_{2n}$ we have the relation

$$\delta_{[\lambda],[\mu]} = \frac{|[\lambda]|}{(2n)!} \sum_{i=1}^{k} \chi^{(i)}(\lambda)\chi^{(i)}(\mu) \tag{3.4}$$

Replacing the sum over $\mathcal{C}_{(2^n)}$ by a sum over $\mathfrak{S}_{2n}$ and substituting (3.4) into (3.2) and (3.3) we obtain the desired expression. $\square$

Theorem 3.3.1 provides an explicit way to compute meandric and semi-meandric numbers but is of little practical use. To determine, for instance, that there is a unique semi-meander of order 2 would involve summing over

6000 terms, since for $\mathfrak{S}_4$ there are two conjugacy classes with 3 cycles, 24 elements, and 5 conjugacy classes.

In fact the expression in terms of characters is a direct translation of the algebraic characterization and does not immediately provide any additional insight into the structure of the problem. It preserves the parts of the combinatorial construction of semi-meanders as identifiable pieces. Considering the expression for $\overline{M}_n$, for example, we begin with a sum over all permutations in $\mathfrak{S}_{2n}$. Summing over h restricts the sum to products of n disjoint transpositions, while summing over i further restricts the sum to include only those elements of $\mathcal{C}_{(2^n)}$ that correspond to arch configurations. Taking the final sum over j leaves only semi-meandric systems that are connected.

The utility of the expression comes from the fact that it provides a bridge from combinatorics to representation theory. It may be possible to expand the characters in terms of symmetric functions, or even find an alternate expansion for the δ-function, although, to date, no significant advances have been made in this direction.

Chapter 4

The Matrix Model

The method of matrix integrals can be used to transform the meander problem into an analytic question. By selecting an appropriate Gaussian measure on the space of Hermitian matrices, the average value of a matrix expression can be discretised so that the non-vanishing terms correspond to a class of decorated ribbon graphs from which meanders can be isolated. The construction used is from [11].

4.1 Meanders as Ribbon Graphs

We begin with an informal definition of a ribbon graph.

Definition 4.1.1. *A* ribbon graph *is an object obtained from a graph, by embedding it in a locally orientable surface, thickening the edges into ribbons, and deleting the faces. The embedding imposes a cyclic ordering of the edges around each vertex.*

A ribbon graph preserves the face boundaries of its embedding. Consequently, the original graph and embedding can be recovered by stitching an open disc along each face boundary, and contracting each ribbon into an edge.

By considering the one-point compactification of the plane, a meander has a natural interpretation as a graph embedded in the sphere: the vertices are intersections between the curves, and the edges are derived from segments of the two curves, with an additional edge joining the extreme vertices along the line. These edges are decorated with two colours, to distinguish those associated with the closed curve from those associated with the line. The encoding is made reversible by rooting the graph. The first vertex along the line is the root vertex, and the edge joining this vertex to the second vertex along the line is the root edge. We produce a decorated ribbon graph by thickening the edges, and preserving their colouring. Figure 4.1 illustrates this encoding. For the purpose of enumeration, we define a

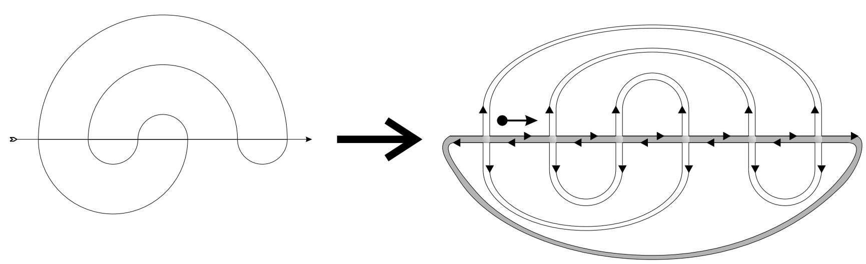


Figure 4.1: Encoding a closed meander as a ribbon graph

class of ribbon graphs that generalizes the ribbon graphs that correspond to meanders.

Definition 4.1.2. *The class $\mathcal{R}$ is the class of oriented 4-regular ribbon graphs on labelled vertices, with edges divided into two classes, such that around every vertex the edges alternate between the two classes. For each vertex, one of the edges of the second class is designated as up.*

The graph induced on a ribbon graph of class $\mathcal{R}$ by taking only the edges of one class is 2-regular. Thus the two classes of edges induce two collections of disjoint cycles. We have the following lemma for identifying meanders within the class $\mathcal{R}$.

Lemma 4.1.3. *Meanders of order n are in $4n$ to $(2n)! \cdot 2^{2n}$ correspondence with graphs in $\mathcal{R}$ on $2n$ vertices that are genus zero and have exactly two cycles induced by the partitioning of the edges, one of each class.*

Proof. Beginning with a ribbon graph we obtain an unlabelled graph by discarding the labels and orientations on the vertices. This results in a $(2n)! \cdot 2^{2n}$ to one correspondence. A meander is obtained from such an unlabelled graph by designating a root vertex and picking one of the edges of the second class incident to it as the root edge. Picking the root vertex and edge accounts for the additional factor of $2n \cdot 2 = 4n$. $\square$

Guided by Lemma 4.1.3, we will use $\mathcal{R}_m$ to denote the elements of $\mathcal{R}$ with m vertices, and consider the generating series

$$Z(s, q, N) = \frac{1}{N^2} \sum_{m \geq 1} \frac{(-1)^m}{m!} \frac{s^m}{N^m} \sum_{G \in \mathcal{R}_m} N^{p(G)} q^{r(G)}, \tag{4.1}$$

where $p(G)$, and $r(G)$ denote, respectively, the number of faces of G, and the number of cycles induced by the edge colouring, in G. By further decorating the elements of $\mathcal{R}$, assigning each face a label from the set $\{1, 2, \ldots, N\}$, and assigning each of the cycles induced by the partitioning of the edges a label from the set $\{1, 2, \ldots, q\}$, we obtain the term $N^{p(G)} q^{r(G)}$ as the number of distinct decorations of the map G.

Associated with each vertex are the labels of the four faces with which it is incident, labelled i_1, i_2, i_3, and i_4 in cyclic order, and two labels on the edges, k for the edges of the first class, l for the edges of the second class. Figure 4.2 gives such a labelled neighbourhood.

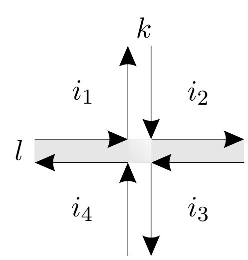


Figure 4.2: The neighbourhood of a vertex

We assign to each half-edge a variable carrying its labelling information, and assign to each vertex the product of these variables. In clockwise order starting with the upper branch, the branches of Figure 4.2 are assigned the variables $h^{(k)}_{i_1i_2}$, $g^{(l)}_{i_2i_3}$, $h^{(k)}_{i_3i_4}$, and $g^{(l)}_{i_4i_1}$, and the vertex is assigned the monomial

$$h^{(k)}_{i_1i_2}g^{(l)}_{i_2i_3}h^{(k)}_{i_3i_4}g^{(l)}_{i_4i_1},$$

where the h's and g's are triply indexed variables.

Proposition 4.1.4. *If, for $1 \le k \le q$, we let $\mathbf{H}_k$ denote the $N \times N$ matrix $\left(h^{(k)}_{ij}\right)$ and $\mathbf{G}_k$ denote the $N \times N$ matrix $\left(g^{(k)}_{ij}\right)$, then the monomials associated with all possible vertex neighbourhoods are enumerated by*

$$\operatorname{tr} \sum_{k,l=1}^{q} (\mathbf{H}_k\mathbf{G}_l)^2.$$

Proof. The sum of the monomials associated with all possible labellings is

$$\sum_{k,l=1}^{q} \sum_{i_1,i_2,i_3,i_4=1}^{N} h^{(k)}_{i_1i_2}g^{(l)}_{i_2i_3}h^{(k)}_{i_3i_4}g^{(l)}_{i_4i_1} = \operatorname{tr} \sum_{k,l=1}^{q} \mathbf{H}_k\mathbf{G}_l\mathbf{H}_k\mathbf{G}_l \qquad \square$$

Our strategy is to enumerate all possible decorated ribbon graphs of class $\mathcal{R}$ on m vertices by picking out all the graphs with a given m-tuple of vertex neighbourhoods, for all possible m-tuples. Given a collection of vertex neighbourhoods, we can describe a ribbon graph by specifying a coupling (a complete matching) of the branches of the vertex neighbourhoods into the $2m$ pairs of branches that are connected to form edges. Such a coupling must be consistent with the labelling, and not introduce any twists into the ribbons, as depicted in Figure 4.3.

Thus, to enumerate all valid maps for a given m-tuple of vertex neighbourhoods, we need to determine the number of consistent branch couplings for that m-tuple. In terms of the monomials, a coupling is consistent if every variable $h^{(k)}_{ij}$ is paired with $h^{(k)}_{ji}$ and every variable $g^{(k)}_{ij}$ is paired with $g^{(k)}_{ji}$. To deal with these pairings, we use some results from probability theory about Gaussian measures.

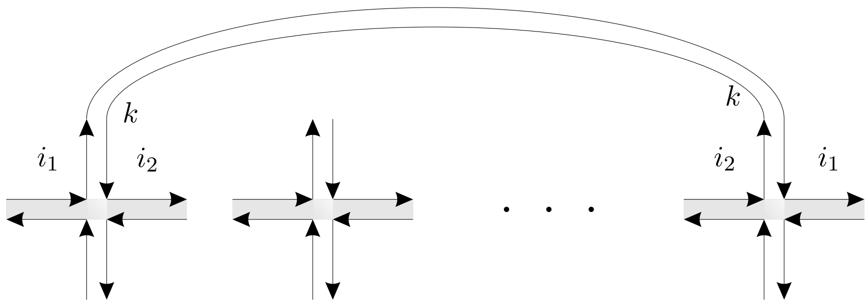


Figure 4.3: A coupling

4.2 Gaussian Measures

If $\mathbf{B}$ is a positive definite matrix giving a quadratic form on $\mathbb{R}^n$, then the measure

$$\mathrm{d}\mu(x) = (2\pi)^{-n/2}(\det \mathbf{B})^{1/2} \exp\left(-\tfrac{1}{2}x^T\mathbf{B}x\right) \mathrm{d}x,$$

is the *Gaussian measure* on $\mathbb{R}^n$ associated with $\mathbf{B}$, where x is an n component column vector, and $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^n$. The measure $\mathrm{d}\mu$ is a probability measure, that is, the measure of the entire space is 1. For a function f we use the notation

$$\langle f \rangle = \int_{\mathbb{R}^n} f(x)\mathrm{d}\mu(x)$$

to denote the average value of f with respect to the measure $\mathrm{d}\mu$. We note that, since $\mathrm{d}\mu(x) = \mathrm{d}\mu(-x)$ for all x in $\mathbb{R}^n$, the average value of any odd degree monomial with respect to this measure is zero. The average value of a degree two monomial is given by the following lemma.

Lemma 4.2.1. *If $x_1, x_2, \ldots, x_n$ are the coordinate functions on $\mathbb{R}^n$, with respect to the measure $\mathrm{d}\mu$ associated with the matrix $\mathbf{B}$, then*

$$\langle x_i x_j \rangle = a_{ij},$$

for all $1 \leq i, j \leq n$, where $\mathbf{A} = (a_{ij}) = \mathbf{B}^{-1}$.

A second lemma allows us to calculate the average value of an arbitrary monomial.

Lemma 4.2.2 (Wick's Formula). *If $f_1, f_2, \ldots, f_{2m}$ are linear functions of x, not necessarily distinct, then*

$$\langle f_1 f_2 \cdots f_{2m} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \cdots \langle f_{p_m} f_{q_m} \rangle ,$$

where the sum is taken over the $(2m-1)!!$ couplings of $\{1, 2, \ldots, 2m\}$ into pairs $\{p_i, q_i\}$. Such a partition is referred to as a Wick coupling.

By the linearity of the integral operator, $\langle \cdot \rangle$ is a linear operator, so these lemmas are sufficient for evaluating the average value of any polynomial.

We now consider a specific Gaussian measure on the space $\mathcal{H}_N$ of $N \times N$ Hermitian matrices. Matrices in this space can be coordinatized by N^2 real numbers: for the matrix $\mathbf{H} = (h_{ij})$, we use the coordinates $x_{ij} = \operatorname{Re} h_{ij}$ for $i \leq j$, and $y_{ij} = \operatorname{Im} h_{ij}$ for $i < j$. In these coordinates, $\mathcal{H}_N$ is isomorphic to $\mathbb{R}^{N^2}$. Note that for $\mathbf{H} \in \mathcal{H}_N$,

$$\operatorname{tr} \mathbf{H}^2 = \sum_{i,j=1}^{N} h_{ij} h_{ji} = \sum_{i,j=1}^{N} h_{ij} \overline{h_{ij}} = \sum_{i=1}^{N} x_{ii}^2 + 2 \sum_{i<j} (x_{ij}^2 + y_{ij}^2)$$

is a positive definite quadratic form on $\mathcal{H}_N$. With respect to the coordinates x_{ij} and y_{ij}, the matrix for the quadratic form is diagonal, with N 1's corresponding to the coordinates x_{ii}, and $N^2 - N$ 2's corresponding to the coordinates x_{ij} and y_{ij} with $i \neq j$. It thus has determinant 2^{N^2-N}. We can construct the associated Gaussian measure $\mathrm{d}\mu(\mathbf{H})$ on $\mathcal{H}_N$ as

$$\mathrm{d}\mu(\mathbf{H}) = (2\pi)^{-N^2/2} 2^{(N^2-N)/2} \exp\left(-\tfrac{1}{2} \operatorname{tr} \mathbf{H}^2\right) \mathrm{d}v(\mathbf{H}),$$

where

$$\mathrm{d}v(\mathbf{H}) = \prod_{i=1}^{N} \mathrm{d}x_{ii} \prod_{1 \leq i < j \leq N} \mathrm{d}x_{ij} \mathrm{d}y_{ij}$$

is the usual Lebesgue measure on the space. We view the matrix entries h_{ij} as linear functions of the coordinates x_{ij} and y_{ij}, and obtain the following.

Lemma 4.2.3. *With respect to the measure* $\mathrm{d}\mu$ *associated with the quadratic form* $\operatorname{tr}\mathbf{H}^2$,

$$\langle h_{ij}h_{kl}\rangle = \delta_{i,l}\delta_{j,k}.$$

Proof. The inverse of the matrix of the quadratic form is diagonal with N 1's and $N^2 - N$ $\frac{1}{2}$'s. So by Lemma 4.2.1 we have,

$$\langle h_{ii}h_{ii}\rangle = \langle x_{ii}^2\rangle = 1$$

$$\langle h_{ji}h_{ij}\rangle = \langle h_{ij}h_{ji}\rangle = \langle h_{ij}\overline{h_{ij}}\rangle = \langle x_{ij}^2 + y_{ij}^2\rangle = \tfrac{1}{2} + \tfrac{1}{2} = 1 \qquad \text{when } i < j.$$

In the remaining cases, when $(i,j) \neq (l,k)$, we get no contribution from diagonal terms and the average is zero. □

To enumerate decorated elements of $\mathcal{R}$ we extend the measure on $\mathcal{H}_N$ to a measure on the space $(\mathcal{H}_N)^{2q}$ of $2q$-tuples $(\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_q, \mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_q)$ of $N \times N$ Hermitian matrices. As a corollary to Lemma 4.2.3 we have the following.

Corollary 4.2.4. *In the space* $(\mathcal{H}_N)^{2q}$ *where* $\mathbf{H}_k = \left(h_{ij}^{(k)}\right)$ *and* $\mathbf{G}_k = \left(g_{ij}^{(k)}\right)$ *for* $1 \le k \le q$*, the following relations hold with respect to the Gaussian measure described above:*

$$\begin{aligned}
\left\langle h_{i_1j_1}^{(k_1)} h_{i_2j_2}^{(k_2)}\right\rangle &= \delta_{k_1,k_2}\delta_{i_1,j_2}\delta_{i_2,j_1},\\
\left\langle g_{i_1j_1}^{(k_1)} g_{i_2j_2}^{(k_2)}\right\rangle &= \delta_{k_1,k_2}\delta_{i_1,j_2}\delta_{i_2,j_1}, \ \ \textit{and}\\
\left\langle h_{i_1j_1}^{(k_1)} g_{i_2j_2}^{(k_2)}\right\rangle &= 0.
\end{aligned}$$

Interpreting the variables as labels on branches of vertex neighbourhoods, as in Section 4.1, we conclude that for branches labelled a and b,

$$\langle ab\rangle = \begin{cases} 1 & \text{if the branches can be paired consistently}\\ 0 & \text{if the branches cannot be paired.}\end{cases}$$

Armed with Wick's Formula, we can give a combinatorial interpretation to the expression

$$\left\langle \left(\operatorname{tr} \sum_{k,l=1}^{q} \mathbf{H}_k \mathbf{G}_l \mathbf{H}_k \mathbf{G}_l \right)^m \right\rangle.$$

By the linearity of $\langle \cdot \rangle$, this expression is a sum over all Wick couplings of the variables involved in all possible m-tuples of vertex neighbourhoods. By the above observation, a coupling contributes 1 to the sum if it can be interpreted as a decorated element of $\mathcal{R}$ and 0 otherwise. We conclude that

$$\left\langle \left(\operatorname{tr} \sum_{k,l=1}^{q} \mathbf{H}_k \mathbf{G}_l \mathbf{H}_k \mathbf{G}_l \right)^m \right\rangle = \sum_{G \in \mathcal{R}_m} N^{p(G)} q^{r(G)}. \tag{4.2}$$

By substituting this expression into (4.1), we obtain the expression

$$\begin{aligned} Z(s,q,N) &= \frac{1}{N^2} \sum_{m \geq 0} \frac{(-1)^m}{m!} \frac{s^m}{N^m} \left\langle \left(\operatorname{tr} \sum_{k,l=1}^{q} \mathbf{H}_k \mathbf{G}_l \mathbf{H}_k \mathbf{G}_l \right)^m \right\rangle \\ &= \frac{1}{N^2} \left\langle \exp\left(-\frac{s}{N} \operatorname{tr} \sum_{k,l=1}^{q} \mathbf{H}_k \mathbf{G}_l \mathbf{H}_k \mathbf{G}_l \right) \right\rangle \\ &= \frac{1}{N^2} \int_{(\mathcal{H}_N)^{2q}} \exp\left(-\frac{s}{N} \operatorname{tr} \sum_{k,l=1}^{q} \mathbf{H}_k \mathbf{G}_l \mathbf{H}_k \mathbf{G}_l \right) \prod_{k,l=1}^{q} \mathrm{d}\mu(\mathbf{H}_k) \mathrm{d}\mu(\mathbf{G}_l). \end{aligned} \tag{4.3}$$

Notice, that in this expression, N and q are not indeterminates. The expression holds only positive integers N and q. Using the expression to evaluate $Z(s,q,N)$ at different values of q or N requires evaluating different integrals, but,

$$P_m(q,N) = \sum_{G \in \mathcal{R}_m} N^{p(G)} q^{r(G)} \tag{4.4}$$

is a polynomial of bounded degree in q and N, and can, in principle, be determined by interpolation.

4.3 Recovering Meanders

Expression (4.3) describes a generating series for $\mathcal{R}$ with respect to number of vertices, faces, and induced cycles. A generating series for elements of $\mathcal{R}$ with exactly two induced cycles can be obtained from $Z(s,q,N)$ by considering the coefficient of q^2,

$$Y(s,N) = [q^2]Z(s,q,N).$$

In $Y(s,N)$ a graph G with m vertices and p faces is weighted by a factor of $(-1)^m s^m/(m! \cdot N^{m+2-p})$.

We further specialize to planar graphs by considering the Euler characteristic of the graphs under consideration. Since a 4-regular graph on m vertices has $2m$ edges, such a graph has Euler characteristic $m - 2m + p = p - m$. Now a graph of genus g has Euler characteristic $2 - 2g$, so $m + 2 - p = 2g$. Thus we obtain

$$Y_0(s) = \lim_{N \to \infty} Y(s,N)$$

as the generating series for genus zero elements of $\mathcal{R}$ with exactly two induced cycles. In $Y_0(s)$ a graph G with m vertices is weighted by a factor of $\frac{1}{m!}(-1)^m s^m$. This yields the main theorem for the chapter.

Theorem 4.3.1. *The meandric numbers can be obtained from $Y_0(s)$ by the relation*

$$\frac{2^{2n}}{4n} M_n = [s^{2n}]Y_0(s).$$

Proof. This is an immediate consequence of the preceding observations and Lemma 4.1.3. □

Corollary 4.3.2. *Where $M_n(s)$ is the ordinary generating series for meanders with respect to their order*

$$2sY_0'(s) = M(4s^2).$$

The construction discussed in this chapter does not provide a final solution to the meander problem. In particular, a complete solution would require a method to evaluate the integral in (4.3), a task for which little progress has been made to date. Several techniques exist for evaluating matrix integrals, but none of them are directly applicable in this case.

In general, matrix integrals are evaluated by a change of coordinates, effectively diagonalizing the matrices involved and working over $\mathbb{R}^n$. The difficulty in this case comes from the fact that the matrices $\mathbf{H}_k$ and $\mathbf{G}_l$ are not simultaneously diagonalizable. Unlike the expression $\operatorname{tr}\mathbf{H}^4$, which is used in enumerating 4-regular maps with no added structure, the expression $\operatorname{tr}\mathbf{H}_k\mathbf{G}_l\mathbf{H}_k\mathbf{G}_l$ is not invariant under unitary transformations on $\mathbf{H}_k$. The obvious change of coordinates $\mathbf{H} \to (\mathbf{U}, \mathbf{\Lambda})$, where $\mathbf{U}$ is unitary, $\mathbf{\Lambda}$ is diagonal, and $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}$, is not applicable.

4.4 Another Matrix Model

The integral from (4.3) is not unique in its utility in enumerating meanders. It is possible that some similar integral may carry the same enumerative information but be more susceptible to evaluation.

In particular, it is worth mentioning the matrix integral used by Di Francesco, Golinelli, and Guitter, in [5] and [7]. They consider a different Gaussian measure, one normalized such that $\langle h_{ij}h_{kl}\rangle = \frac{1}{N}\delta_{i,l}\delta_{j,k}$, and consider an expression with an extra parameter: instead of working over $(\mathcal{H}_N)^{2q}$, they define an integral over $(\mathcal{H}_N)^{q_1+q_2}$, separately controlling the number of $\mathbf{H}$ and $\mathbf{G}$ matrices.

The result is an expression that records separately the number of cycles of each class. A restriction to connected graphs is obtained by taking a logarithm. Interpreting the resulting expression as a polynomial in q_2, and taking the limit as q_2 approaches zero, further restricts the expression to graphs with only a single induced cycle of the second class. The genus zero case is recovered as before. This approach has the advantage that it has the potential to be used for determining the number of meandric systems with respect to both order and number of components, since the number of cycles of the second class is recorded.

To date, the integral obtained by Di Francesco *et al.* has been no more susceptible to existing evaluation techniques than the integral from (4.3), but, in [7], Di Francesco *et al.* note that it is suggestive of a two-dimensional conformal field theory, and using this similarity, and universality principles from statistical mechanics, they conjecture the exact values

$$\alpha = \frac{\sqrt{29}}{12}\left(\sqrt{29}+\sqrt{5}\right) \qquad \text{and} \qquad \overline{\alpha} = 1 + \frac{\sqrt{11}}{24}\left(\sqrt{29}+\sqrt{5}\right)$$

in the asymptotic approximations to M_n and $\overline{M}_n$ (2.4) and (2.4′). Numerical estimates of α and $\overline{\alpha}$, obtained by Jensen and Guttmann in [10] through the method of differential approximants, disagree with the conjectured values.

Chapter 5

The Temperley-Lieb Algebra

In this chapter, we shift attention to meandric systems (recall Definition 2.2.1). Upper and lower arch configurations are encoded as strand diagrams, which are in turn interpreted as elements of the Temperley-Lieb algebra. Under this encoding, the number of components of a meandric system can be recovered by evaluating a bilinear form on its upper and lower configurations. Choosing an appropriate basis for the algebra, allows us to write the meandric polynomials in terms of the Gram matrix of this bilinear form.

5.1 Strand Diagrams

Before defining the Temperley-Lieb algebra, we introduce *strand diagrams*, which can be used to pictorially represent the algebra. Strand diagrams provide a combinatorial presentation of the algebra and a vehicle for linking the algebra to the meander problem.

Definition 5.1.1. *A* strand diagram *of order n, is a configuration of n pairwise non-intersecting curves, called strands, in an open disc, that connect $2n$ endpoints on the boundary of the disc. These points are labelled cyclically by $\{1, 2, \ldots, 2n\}$. Two strand diagrams are equivalent if there is a homeomorphism from one to the other that respects the labelling of the endpoints.*

We represent strand diagrams by rectangles, with the labels $\{1, 2, \ldots, n\}$ running down the left side of the rectangle, and the labels $\{n+1, n+2, \ldots, 2n\}$ running up the right side. Figure 5.3 gives a strand diagram of order 6.

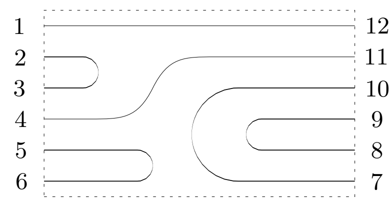


Figure 5.1: A strand diagram of order 6

The connection between strand diagrams and meandric systems comes from the following lemma, which is illustrated in Figure 5.2.

Lemma 5.1.2. *There is a natural bijection between arch configurations of order n and strand diagrams of order n.*

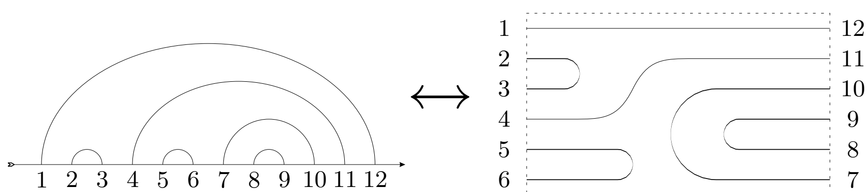


Figure 5.2: A bijection between strand diagrams and arch configurations

Proof. The arches of an arch configuration of order n are pairwise non-intersecting, and connect $2n$ points on the boundary of the upper half-plane. To obtain a strand diagram from an arch configuration, it is sufficient to label the basepoints according to the orientation of the line. The labelling of the basepoints makes the inverse unique. $\square$

5.2 The Temperley-Lieb Algebra

We now define the Temperley-Lieb algebra and develop some of its properties. The reader is referred to [3] for a more complete treatment.

Definition 5.2.1. *The* Temperley-Lieb algebra *of order n in the indeterminate q, denoted $\mathfrak{TL}_n(q)$, is a free additive algebra over $\mathbb{C}(q)$ with multiplicative generators 1, e_1, e_2, $\ldots$, e_{n-1} subject to the relations:*

$$\mathsf{e}_i^2 = q\,\mathsf{e}_i \qquad \textit{for } i = 1, 2, \ldots, n-1 \tag{R1}$$

$$\mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i \qquad \textit{if } |i-j| > 1 \tag{R2}$$

$$\mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i = \mathsf{e}_i \qquad \textit{for } i = 1, 2, \ldots, n-1 \tag{R3}$$

where 1 is the multiplicative identity.

Pictorially, the generators of $\mathfrak{TL}_n(q)$ are represented as the strand diagrams given in Figure 5.3. In this representation, multiplication is by

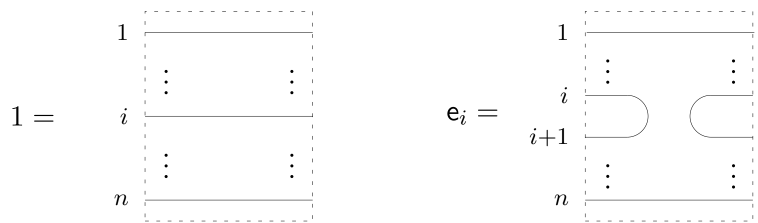


Figure 5.3: The multiplicative generators of $\mathfrak{TL}_n(q)$

concatenation. To construct the product ef from strand diagrams e and f:

1. identify the right edge of e with the left edge of f,
2. for $1 \leq i \leq n$ identify the endpoint labelled i in f with the endpoint labelled $n - i + 1$ in e,
3. delete every closed loop introduced by concatenation, accounting for each loop by replacing it with a multiplicative factor of q.

Figure 5.4 shows the relations (R1), (R2), and (R3) under this representation. Each relation equates a pair of homeomorphic strand diagrams, so products in $\mathfrak{TL}_n(q)$ respect the concatenation product of strand diagrams.

Proposition 5.2.2. *Using the pictorial representation, strand diagrams of order n form a basis for $\mathfrak{TL}_n(q)$ as a $\mathbb{C}(q)$-vector space, and $\mathfrak{TL}_n(q)$ has dimension C_n. We call this basis $\mathcal{B}_1$.*

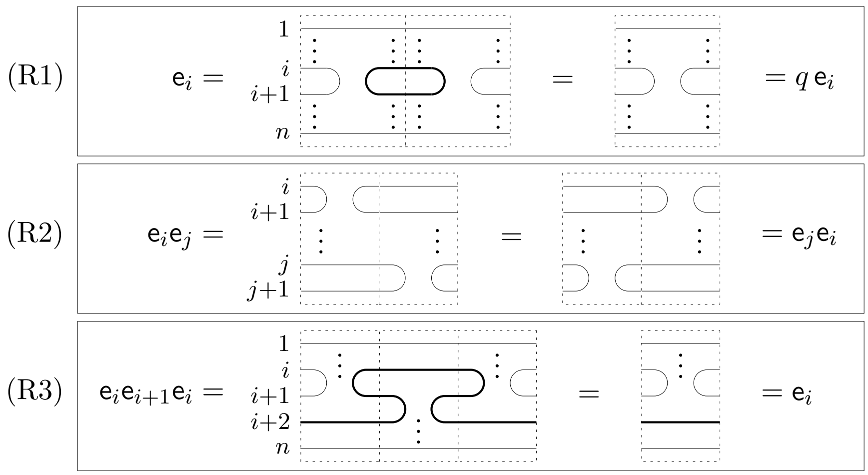


Figure 5.4: A pictorial representation of (R1), (R2), and (R3)

Proof. Using the pictorial representation, every monomial in generators of $\mathfrak{TL}_n(q)$ can be represented as a strand diagram of order n, and two monomials that are equivalent under (R1), (R2), and (R3) are represented by equivalent strand diagrams. Thus, to verify that strand diagrams form a basis for $\mathfrak{TL}_n(q)$, we need only show that every strand diagram can be factored into a product of the multiplicative generators. The dimension of the algebra is then a consequence of Lemma 5.1.2.

We describe a construction for factoring an arbitrary strand diagram of order n into a product of multiplicative generators of $\mathfrak{TL}_n(q)$. Appealing to Lemma 5.1.2, we begin by representing the strand diagram as an arch configuration of order n. The arch configuration is then further encoded as a *path diagram*, a graph in $\mathbb{R}^2$ on the vertices $v_i = (i, h_i)$ for $0 \leq i \leq 2n$, where h_i is the number of arches passing over the midpoint between the i-th and $(i+1)$-st basepoints and, by convention, $h_0 = h_{2n} = 0$.

Example 5.2.3. *The strand diagram from Figure 5.3 is represented as the arch configuration Figure 5.5* (a)*. Encoding this as a path diagram produces Figure 5.5* (b)*.*

The value h_n is the number of strands passing from left to right in the

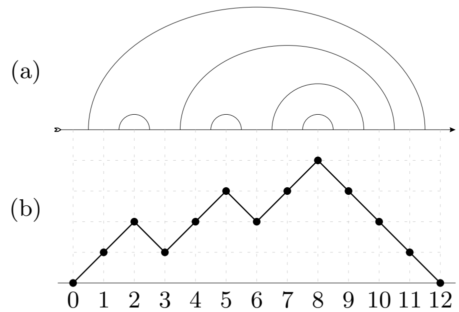


Figure 5.5: Encoding an arch configuration as a path diagram

original strand diagram and determines the (two-sided) ideal it principally generates. We introduce the notation that

$$\mathfrak{I}_n^k = \mathsf{e}_1\mathsf{e}_3 \cdots \mathsf{e}_{n-k-1} \tag{5.1}$$

is the canonical generator for the ideal in $\mathfrak{TL}_n(q)$ containing all strand diagrams with at most k strands passing from left to right.

We decompose the path diagram into a *base path*, the path representation of $\mathfrak{I}_n^{h_n}$, and a collection of boxes that, when stacked on the base path, give the contour of the path diagram. The boxes are labelled according to their positions: a box is labelled by e_i if it is either to the left of the midpoint and centered over x-coordinate i, or to the right of the midpoint and centered over x-coordinate $2n-i$.

Example 5.2.4. *Decomposing Figure 5.5* (b) *produces Figure 5.6, with the base path indicated by a thickened line.*

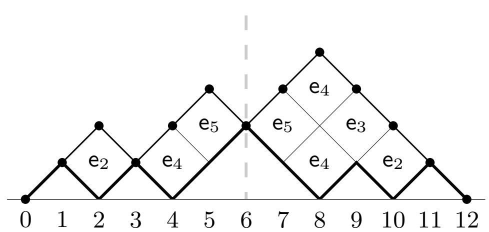


Figure 5.6: Decomposing a path diagram

A factorization of the strand diagram can be read directly from this box decomposition. The strand diagram is recovered by left multiplying $\mathfrak{I}_n^k$ by the labels of the boxes to the left of the midpoint, read in order of increasing height, and right multiplying the result by the labels of the boxes to the right of the midpoint, read in order of increasing height. Since the labels of two boxes at the same height commute, boxes at the same height can be read in any order.

This completes the proof. $\square$

The construction provided does not always produce the most compact factorization of a strand diagram, but has the benefit of emphasizing the role played by the ideals principally generated by $\mathfrak{I}_n^k$.

Example 5.2.5. *Working from Figure 5.6, we have the base path* $\mathfrak{I}_6^2 = \mathsf{e}_1\mathsf{e}_3$ *and obtain the factorization* $S = \mathsf{e}_5(\mathsf{e}_2\mathsf{e}_4)(\mathsf{e}_1\mathsf{e}_3)(\mathsf{e}_2\mathsf{e}_4)(\mathsf{e}_3\mathsf{e}_5)\mathsf{e}_4$*, with commuting generators grouped by parentheses. This is illustrated in Figure 5.7* (a)*. The term corresponding to the base path is indicated by a thickened line. This factorization can be simplified by applying* (R2) *and* (R3) *to obtain*

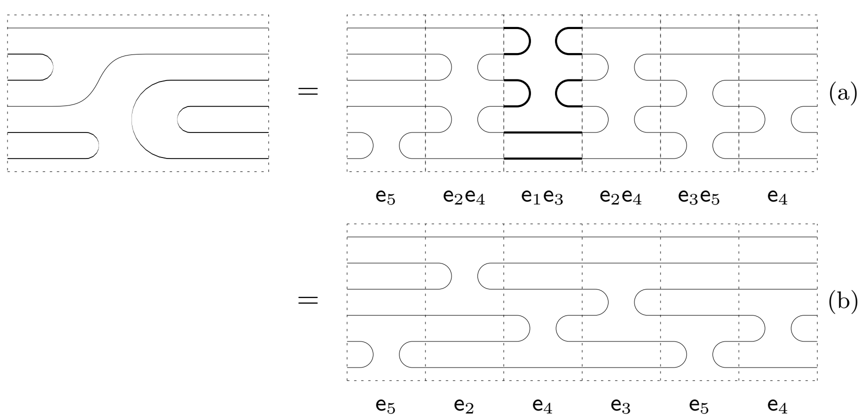


Figure 5.7: A factorization of a strand diagram

$S = \mathsf{e}_5\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5\mathsf{e}_4$*, as given in Figure 5.7* (b)*.*

We now have an encoding of arch configurations of order n as elements of $\mathfrak{TL}_n(q)$. By representing meandric systems as ordered pairs of arch configurations, we naturally extend this encoding to meandric systems: the element $(\mathsf{e}, \mathsf{f}) \in \mathfrak{TL}_n(q) \times \mathfrak{TL}_n(q)$ encodes the meandric system with upper arch configuration encoded as e, and lower arch configuration encoded as f. To determine the number of components of a meandric system so encoded, we introduce a trace on the algebra.

We think of $\big(\mathfrak{TL}_1(q), \mathfrak{TL}_2(q), \dots\big)$ as a sequence of nested algebras. The algebra $\mathfrak{TL}_n(q)$ is embedded in $\mathfrak{TL}_{n+1}(q)$ by the map

$$\begin{aligned} \varphi\colon \mathfrak{TL}_n(q) &\to \mathfrak{TL}_{n+1}(q) \\ \mathsf{e}_i &\mapsto \mathsf{e}_i, \end{aligned}$$

extended as a homomorphism. In terms of strand diagrams, the map adds an additional strand at the bottom of the diagram, and relabels the endpoints appropriately.

Definition 5.2.6. *A family of functions* $\operatorname{tr}_k\colon \mathfrak{TL}_k(q) \to \mathbb{C}(q)$ *for* $k \geq 1$ *is called a* Markov trace *if it satisfies:*

$$\operatorname{tr}_k \text{ is a linear functional,} \tag{T1}$$

$$\operatorname{tr}_k(\mathsf{ef}) = \operatorname{tr}_k(\mathsf{fe}) \text{ for all } \mathsf{e}, \mathsf{f} \in \mathfrak{TL}_k(q), \text{ and} \tag{T2}$$

$$\operatorname{tr}_{k+1}(\mathsf{e}\mathsf{e}_k) = \operatorname{tr}_k(\mathsf{e}) \text{ if } \mathsf{e} \in \langle 1, \mathsf{e}_1, \dots, \mathsf{e}_{k-1} \rangle. \tag{T3}$$

A function tr_k *satisfying* (T1) *and* (T2) *is called a* trace*.*

Notice that $\operatorname{tr}_{k+1}(\mathsf{e}\mathsf{e}_k\mathsf{f}) = \operatorname{tr}_{k+1}(\mathsf{f}\mathsf{e}\mathsf{e}_k) = \operatorname{tr}_k(\mathsf{fe}) = \operatorname{tr}_k(\mathsf{ef})$ so we could replace (T3) in the definition by

$$\operatorname{tr}_{k+1}(\mathsf{e}\mathsf{e}_k\mathsf{f}) = \operatorname{tr}_k(\mathsf{ef}) \text{ if } \mathsf{e}, \mathsf{f} \in \langle 1, \mathsf{e}_1, \dots, \mathsf{e}_{k-1} \rangle \text{ for } k \geq 1. \tag{T3$'$}$$

A proof that there is a unique Markov trace on $\mathfrak{TL}_k(q)$, up to a multiplicative factor, is given in [3, Appendix A]. As a consequence, a Markov trace is completely determined by any non-zero evaluation. To give a combinatorial interpretation to this trace, we introduce the closure of a strand diagram.

Definition 5.2.7. *The* closure *of a strand diagram S, of order n, is obtained from S by placing the diagram on a cylinder, and identifying the endpoint i with the endpoint $n-i+1$ for each i in the range $1 \leq i \leq n$. Pictorially, this is represented by connecting the endpoints around the outside of the disc, in a planar fashion. Figure 5.8 gives a strand diagram and its closure.*

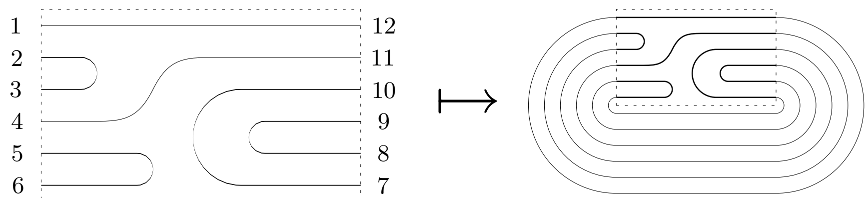


Figure 5.8: A strand diagram and its closure

The closure of a strand diagram consists of a collection of loops. We use $\#_{\text{loop}}(S)$ to denote the number of loops in the closure of S.

Proposition 5.2.8. *The family of functions* tr_k *defined on strand diagrams by*

$$\mathrm{tr}_k(S) = q^{\#_{\text{loop}}(S)}, \tag{5.2}$$

and extended linearly to $\mathfrak{TL}_k(q)$ *is the Markov trace such that* $\mathrm{tr}_1(1) = q$.

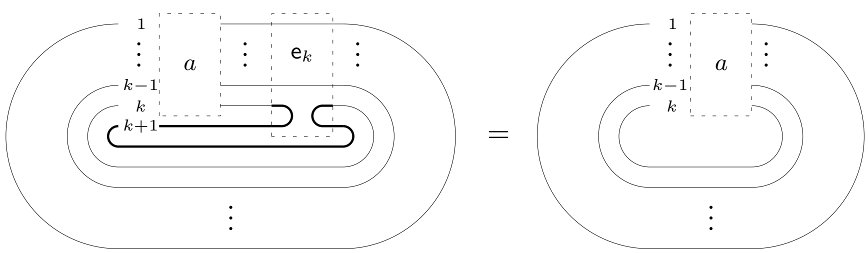


Figure 5.9: tr_k satisfies (T3)

Proof. Since tr_k is linear, we need only verify (T1), (T2), and (T3) for monomials. Condition (T1) is satisfied by definition. Condition (T2) is true since the closure of ef and the closure of fe are obtained by identifying the same pairs of endpoints in e and f; every loop is either deleted and counted by a factor of q by (R1), or counted as a factor of q by $\#_{\text{loop}}(\cdot)$. Condition (T3) is illustrated in Figure 5.9; the portion of the curve indicated by a thickened line can be contracted without altering the number of loops. $\square$

The Markov property of tr_k provides a way to recursively evaluate the function. Consider the following examples.

Example 5.2.9. *We calculate the number of components in the closure of* $S = \mathsf{e}_5\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5\mathsf{e}_4$.

$$\begin{aligned} q^{\#_{\text{loop}}(S)} &= \text{tr}_6(\mathsf{e}_5\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5\mathsf{e}_4) = \text{tr}_6(\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5\mathsf{e}_4\mathsf{e}_5) = \text{tr}_6(\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3\mathsf{e}_5) \\ &= \text{tr}_5(\mathsf{e}_2\mathsf{e}_4\mathsf{e}_3) = \text{tr}_4(\mathsf{e}_2\mathsf{e}_3) = \text{tr}_3(\mathsf{e}_2) = \text{tr}_2(1) = q^2 \end{aligned}$$

We conclude that the closure of S consists of 2 *loops, in agreement with Figure 5.8.*

Example 5.2.10. *By using* (R3) *to introduce additional generators, we can recursively evaluate* tr_{k+1} *in terms of* tr_k *even when* e_k *does not occur as a factor. Consider* $\text{tr}_4(\mathsf{e}_1) = \text{tr}_4(\mathsf{e}_1\mathsf{e}_2\mathsf{e}_1) = \text{tr}_4(\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3\mathsf{e}_2\mathsf{e}_1) = \text{tr}_3(\mathsf{e}_1\mathsf{e}_2\mathsf{e}_2\mathsf{e}_1)$, *for example.*

Example 5.2.11. *From the combinatorial interpretation of* tr_k, *we can easily verify that* $\text{tr}_k(1) = q^k$. *This can also be verified inductively, using* (T3) *to introduce a multiplicative generator that can be expanded.*

$$\begin{aligned} \text{tr}_k(1) &= \text{tr}_{k+1}(\mathsf{e}_k) = \text{tr}_k(\mathsf{e}_k\mathsf{e}_{k-1}\mathsf{e}_k) = q\,\text{tr}_k(\mathsf{e}_{k-1}\mathsf{e}_k) \\ &= q\,\text{tr}_k(\mathsf{e}_{k-1}) = q\,\text{tr}_{k-1}(1) \end{aligned}$$

Proposition 5.2.12. *Semi-meandric numbers can be expressed in terms of the trace function and the basis* $\mathcal{B}_1$ *through the expression*

$$\overline{M}_n = [q] \sum_{\mathsf{f} \in \mathcal{B}_1} \text{tr}_n\,\mathsf{f}. \tag{5.3}$$

Proof. By Lemma 5.1.2, we take the sum to be over all arch configurations of order n. An arch configuration a with corresponding strand diagram S

contributes the term $\mathrm{tr}_n(S) = q^{\#_{\mathrm{loop}}(S)}$. Since the closure of S and the semi-meandric system with a as an upper configuration are obtained by connecting the same endpoints, the number of components in the semi-meandric system with a as an upper configuration is $\#_{\mathrm{loop}}(S)$. Thus an arch configuration contributes the term q precisely if the corresponding semi-meandric system is connected. $\square$

In order to make a similar statement relating the meandric numbers to the trace function, we define the transpose f^t of $\mathsf{f} \in \mathfrak{TL}_n(q)$.

Definition 5.2.13. *The* transpose *function* ${}^t\colon \mathfrak{TL}_n(q) \to \mathfrak{TL}_n(q)$ *is defined on the multiplicative basis by*

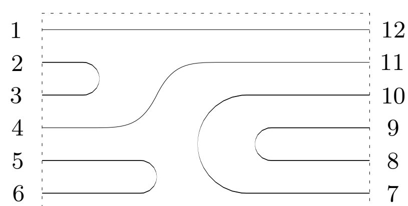
$$\mathsf{e}_i{}^t = \mathsf{e}_i$$

and extended to the whole algebra through

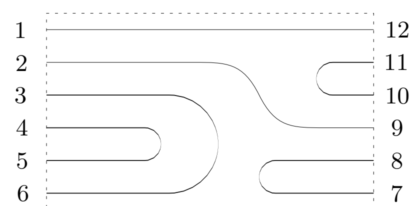
$$(\mathsf{ef})^t = \mathsf{f}^t\mathsf{e}^t \qquad \textit{and} \qquad (\lambda\mathsf{e} + \gamma\mathsf{f})^t = \lambda\mathsf{e}^t + \gamma\mathsf{f}^t$$

for all e, f *in* $\mathfrak{TL}_n(q)$ *and all* λ, γ *in* $\mathbb{C}(q)$.

In terms of strand diagrams, the action of $\cdot^t$ is to reflect the diagram in the vertical axis and to swap the labels i and $2n-i+1$ for each i in the range $1 \le i \le n$. Figure 5.10 gives a strand diagram (a) and its transpose (b).



(a)



(b)

Figure 5.10: A strand diagram (a), and its transpose (b)

Using ^{t} we can define a symmetric bilinear form

$$\langle \cdot, \cdot \rangle_n \colon \mathfrak{TL}_n(q) \times \mathfrak{TL}_n(q) \to \mathbb{C}(q) \tag{5.4}$$

$$(\mathsf{e}, \mathsf{f}) \mapsto \mathrm{tr}_n(\mathsf{ef}^t) \tag{5.5}$$

with the property that if e and f are the representation of arch configurations a and b, then $\langle \mathsf{e}, \mathsf{f} \rangle = q^{c(a,b)}$, where $c(a,b)$ is the number of components in the meandric system with a as its upper configuration and b as its lower configuration.

This last claim is verified by noting that the closure of ef^t is obtained by identifying the endpoint i in e with the endpoint i in f for every i in the range $1 \leq i \leq 2n$. This is precisely the identification involved in creating the meandric system with a as its upper configuration and b as its lower configuration. So the number of components in (a,b) is $\#_{\text{loop}}(\mathsf{ef}^t)$, and we obtain the expression,

$$m_n(q) = \sum_{k=1}^{n} M_n^{(k)} q^k = \sum_{\mathsf{e},\mathsf{f} \in \mathcal{B}_1} \langle \mathsf{e}, \mathsf{f} \rangle_n, \tag{5.6}$$

for the n-th meandric polynomial, since, by Lemma 5.1.2, the sum can be taken to be over all meandric systems of order n. We summarize the form $\langle \cdot, \cdot \rangle_n$ with its *Gram matrix*, $\mathbf{M}_n(q)$, a $C_n \times C_n$ matrix such that

$$\big(\mathbf{M}_n(q)\big)_{ij} = \langle \mathsf{a}_i, \mathsf{a}_j \rangle_n,$$

where $\mathcal{B}_1$ is the ordered basis $(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_{C_n})$, and obtain the following theorem.

Theorem 5.2.14. *The meandric polynomials can be expressed by*

$$m_n(q) = \sum_{i,j=1}^{C_n} \big(\mathbf{M}_n(q)\big)_{ij} = \operatorname{tr}(\mathbf{M}_n(q) \cdot \mathbf{J}_n) = {u_n}^t\, \mathbf{M}_n(q) u_n, \ \textit{and} \tag{5.7}$$

$$m_n(q^2) = \operatorname{tr}\left(\mathbf{M}_n(q)^2\right), \tag{5.8}$$

where u_n is the C_n dimensional column vector of 1's and $\mathbf{J}_n$ is the $C_n \times C_n$ matrix of 1's.

Proof. The expression (5.7) is obtained by rewriting (5.6) in terms of $\mathbf{M}_n(q)$. Since $\mathbf{M}_n(q)$ is symmetric, and every entry is a monic monomial, every term in $\operatorname{tr}(\mathbf{M}_n(q)\mathbf{J}_n)$ appears squared in $\operatorname{tr}\left(\mathbf{M}_n(q)^2\right)$, so (5.8) follows from (5.7). $\square$

Since M_n can be recovered from these expressions through

$$M_n = [q]\, m_n(q) = \left[q^2\right] m_n(q^2),$$

we have reduced the meander problem to the problem of determining the Gram matrix of $\langle\cdot,\cdot\rangle_n$ on $\mathfrak{TL}_n(q)$ with respect to the basis $\mathcal{B}_1$. As with the techniques discussed in previous chapters this is a computationally difficult task.

Example 5.2.15. *With respect to the basis* $\mathcal{B}_1 = \{\mathsf{e}_1, \mathsf{e}_2\mathsf{e}_1, \mathsf{e}_1\mathsf{e}_2, \mathsf{e}_2, 1\}$*, the Gram matrix of* $\langle\cdot,\cdot\rangle_3$ *is*

$$\mathbf{M}_3(q) = \begin{pmatrix} q^3 & q^2 & q^2 & q & q^2 \\ q^2 & q^3 & q & q^2 & q \\ q^2 & q & q^3 & q^2 & q \\ q & q^2 & q^2 & q^3 & q^2 \\ q^2 & q & q & q^2 & q^3 \end{pmatrix}.$$

Using this matrix, we see that $m_3(q) = \operatorname{tr}(\mathbf{M}_3(q)\mathbf{J}_3) = 8q + 12q^2 + 5q^3$*, and recover* $M_3 = [q]m_3(q) = 8$.

Despite the apparent computational difficulty, this approach is promising. The Gram matrix is richly structured and opens the meander problem to the tools of linear algebra. In [6], Di Francesco, *et al.* construct a second basis $\mathcal{B}_2$ for $\mathfrak{TL}_n(q)$ with respect to which the Gram matrix is diagonal, and order the elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that the change of coordinate matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ is triangular. In principle, the meander problem has been reduced to expressing $\mathcal{B}_2$ in terms of $\mathcal{B}_1$, but in practice, only the diagonal entries of the change of coordinate matrix have been determined. This is sufficient to calculate the determinant of $\mathbf{M}_n(q)$ but does not provide a final solution to the problem.

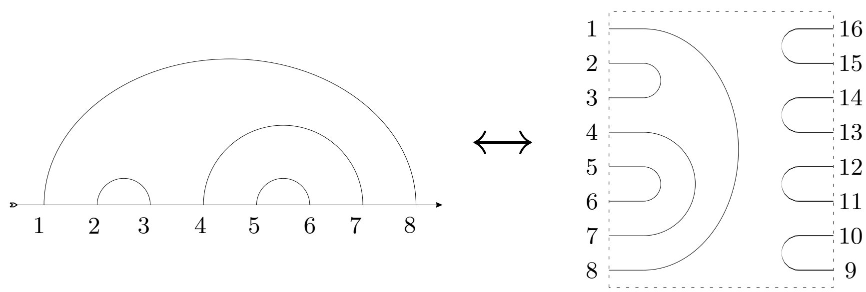


Figure 5.11: A second encoding of arch configurations

Other formulations of the meander problem in terms of the Temperley-Lieb algebra are possible. In particular, Di Francesco suggests in [4], that arch configurations of order n be encoded as monomials in the left ideal $(\mathsf{e}_1\mathsf{e}_3\cdots\mathsf{e}_{2n-1})$ of $\mathfrak{TL}_{2n}(q)$. Under this encoding, an arch configuration of order n is encoded by the strands connecting the labels 1, 2, …, $2n$, as in Figure 5.11. This encoding allows a uniform recursive construction, through box-addition, of the basis elements corresponding to arch configurations, and has the property that if a and b are encoded as e and f, then $\#_{\text{loop}}(\mathsf{e}\mathsf{f}^t) = n+c(a,b)$, with the extra n loops coming from the short arches on the labels $2n{+}1$, $2n{+}2$, …, $4n$.

Chapter 6

Combinatorial Words

Arch configurations have a natural encoding as words in the Dyck language. This chapter describes the extension of this encoding to an encoding of meandric systems as words, and, more generally, deals with enumerative techniques that are based on sequentially considering the intersection points of a meandric system.

6.1 The Encoding

We begin by describing Dyck language. For typographical reasons, when it is inconvenient to use parentheses, we use x and y to denote left and right parentheses. The Dyck language is the language of balanced parentheses on the alphabet $\{x, y\}$, and can be generated by the production rules:

$$\begin{aligned} s &\to \epsilon \\ s &\to xsys, \end{aligned} \tag{6.1}$$

where ϵ is the empty word. The following proposition links the Dyck language to arch configurations.

Proposition 6.1.1. *Arch configurations of order n are in bijective correspondence with Dyck words of length $2n$.*

Proof. To construct an arch configuration of order n from a Dyck word of length $2n$, label $2n$ basepoints sequentially with the letters of the word, and connect each x to its matching y with an arch. The construction is reversed by reading the basepoints sequentially; the first basepoint on an arch is read as x, while the second basepoint on an arch is read as y. $\square$

The bijection from the preceding proof provides an encoding for arch configurations as words over a two-letter alphabet. This encoding is illustrated in Figure 6.1.

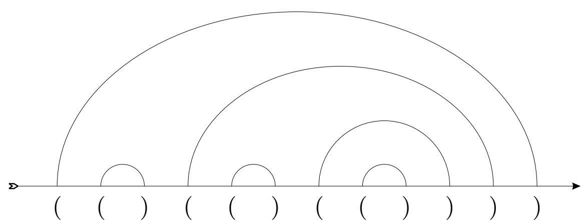


Figure 6.1: An arch configuration as a Dyck word.

By separately encoding the upper and lower configurations of meandric systems as words in the Dyck language, and using the rule,

$$\begin{matrix}(\\(\end{matrix} \to O \qquad \begin{matrix})\\)\end{matrix} \to C \qquad \begin{matrix}(\\)\end{matrix} \to U \qquad \begin{matrix})\\(\end{matrix} \to D,$$

we obtain an encoding of meandric systems as words on the four-letter alphabet $\{C, D, O, U\}$. The language L of meandric systems consists of all words that encode meandric systems under this encoding. A language, of meanders, L' is obtained as the restriction of L to those words that encode meanders.

Having an interpretation of meanders as words, suggests that we consider a sequential decomposition, one that reads meanders from left to right. The i-th *cross section* of a meander is obtained by slicing the canonical representation vertically between bridge i and bridge $i+1$, and recording the position of the line and connectivity of the end points, as determined by the left side of the diagram. The meander '$OODUUDUUDCDC$' is given in

Figure 6.2; dotted lines mark the cross sections. Some of its cross sections are given in Figure 6.3.

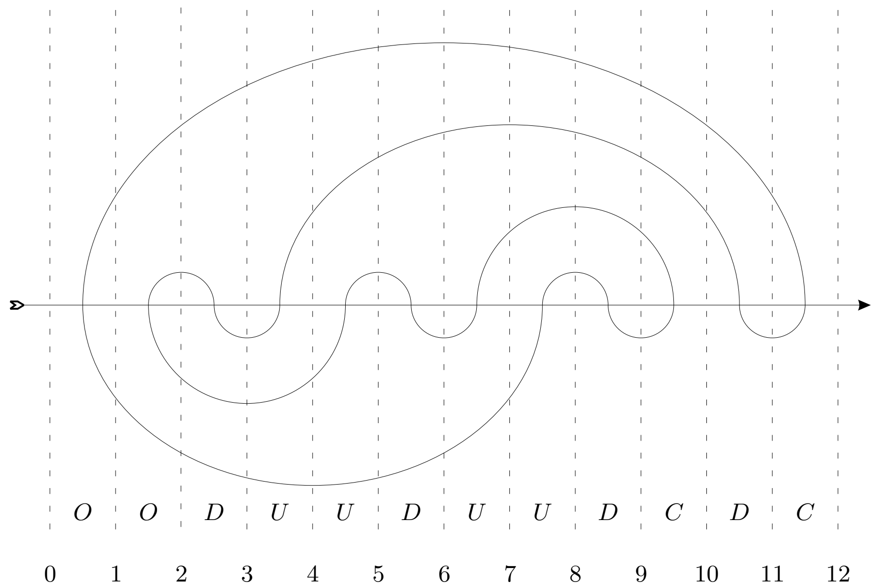


Figure 6.2: The meander '$OODUUDUUDCDC$'

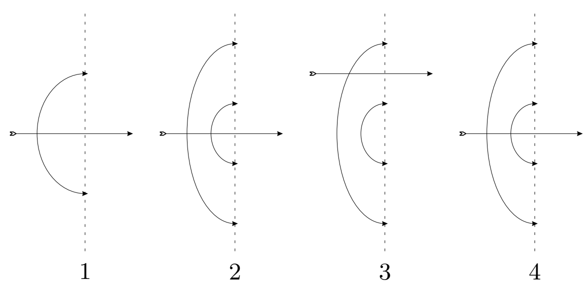


Figure 6.3: Cross sections of '$OODUUDUUDCDC$'

By considering consecutive cross sections, we gain a geometric interpretation for each of the letters in a meandric word. Starting with an empty cross section, the i-th letter specifies the transition that must be applied to the $(i-1)$-st cross section to obtain the i-th cross section: an O opens a new strand around the line, a C closes the strand closest to the line, a U moves the end of a strand up, and a D moves the end of a strand down. These transitions are shown in Figure 6.4. Notice that not every transition can be applied to every cross section: the transition U, for example, can be applied only to a cross section with at least one strand below the line.

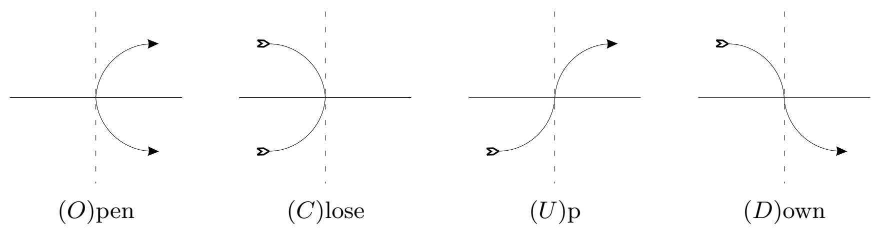


Figure 6.4: The four transitions

In [9], Jensen describes an algorithm, based on cross sections, for computing meandric numbers. His algorithm inductively determines all possible i-th cross sections of meanders and how many sequences of transitions lead to each. The number M_n is the number of sequences of transitions that lead to the empty cross section after $2n$ transitions. When computing only a fixed M_n, the algorithm is optimized to discard cross sections that cannot lead to valid meanders with the remaining number of transitions. In particular, a cross section with more than $2n-i$ strands either above or below the line cannot be the i-th cross section of a meander of order n. Slight modifications can be used to enumerate semi-meanders or meandric systems.

In his analysis, Jensen cites experimental evidence suggesting that the computational complexity of determining the first n meandric numbers using this algorithm grows asymptotically as $\approx 2.5^n$, and that memory use is proportional. In contrast, the tree enumeration approach described in Section 2.5 uses an amount of memory that is polynomial in n, but has time requirements that are proportional to the largest meandric number being calculated, that is $\approx 12.26^n$. In practice, Jensen's algorithm is the most effective method known for computing meandric numbers exactly.

6.2 Irreducible Meandric Systems

Lando and Zvonkin, in [11], successfully use the encoding of meandric systems as words to analyze a class of objects lying strictly between meanders and meandric systems. They call this class *irreducible* meandric systems. It is defined as follows.

Definition 6.2.1. *A meandric system represented by the word W is said to be* irreducible*, if no subword of W represents a meandric system. If W represents an irreducible meandric system, then W is said to be an* irreducible word.

Lando and Zvonkin prove that a meandric system has a unique decomposition into a collection of irreducible meandric systems. In fact the language of meandric systems can be described by the production rules:

$$\begin{aligned} s &\to \epsilon \\ s &\to OsCs \\ s &\to OsUsDsCs \\ &\ \ \vdots \\ s &\to \alpha_1 s \alpha_2 s \cdots \alpha_{2n} s \\ &\ \ \vdots \end{aligned} \tag{6.2}$$

where $\alpha_1\alpha_2\cdots\alpha_{2n}$ denotes a generic irreducible word, and there is one production rule for every irreducible word. By showing that every meandric system has a unique derivation using these rules, they are able to show that the ordinary generating series for meandric systems with respect to order, $B(x)$, and the ordinary generating series for irreducible meandric systems with respect to order, $N(x)$, satisfy the functional equation

$$B(x) = N(xB^2(x)). \tag{6.3}$$

Using analytic techniques, Lando and Zvonkin combine (6.3) and (2.5) to conclude that the radius of convergence of N is

$$\left(\frac{4-\pi}{\pi}\right)^2.$$

This leads to their main result, that meandric numbers satisfy the inequality,

$$M_n < \left(\frac{\pi}{4-\pi}\right)^{2n}, \tag{6.4}$$

for all sufficiently large n values of n.

6.3 Production Rules For Meanders

We do not yet have a collection of production rules for the language L' of meanders that can be used in the manner of (6.1) or (6.2) to produce enumerative results. The substitution rules (the first of which are illustrated in Figure 6.5),

$$\begin{aligned} OUC &\to U & ODC &\to D \\ OUUC &\to UCOU & ODDC &\to DCOD \\ OUUUC &\to UUCDOUU & ODDDC &\to DDCUODD \\ &\vdots & &\vdots \\ OU^{i+2}C &\to U^{i+1}CD^iOU^{i+1} & OD^{i+2}C &\to D^{i+1}CU^iOD^{i+1} \\ &\vdots & &\vdots \end{aligned} \tag{6.5}$$

can be applied reversibly to any meandric system without altering the number of components. Together with the additional substitution rules,

$$UD \to \epsilon \qquad\qquad DU \to \epsilon, \tag{6.6}$$

they form a complete set of substitutions, in the sense that a sequence of substitutions can be used to reduce the word representation of any meander to the word OC. This reduction can be made unambiguous by requiring that each reduction be carried out at the rightmost possible point.

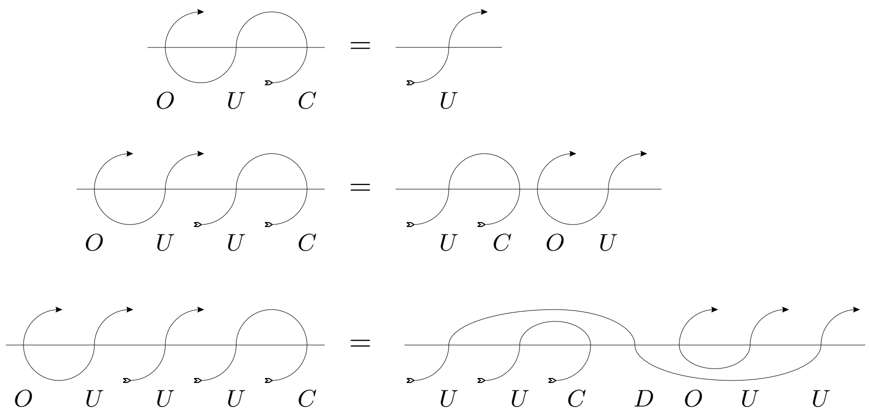


Figure 6.5: Substitution rules for meandric words

In theory, it should be possible to define a set of production rules for producing every meander from the initial string OC. Unfortunately, from the perspective of searching for enumerative utility, only the rules (6.5) can be reversed in a context free fashion. If AB is a word representing a meander, the substitution $AB \rightarrow AUDB$ can be made only if the cross section defined by the prefix A has a strand below the line, and the substitution $AB \rightarrow ADUB$ can be made only if the cross section defined by the prefix A has a strand above the line. The string $OUUDDC$ does not even correspond to a meandric system, despite the fact that it can be produced from OC by twice applying the substitution $\epsilon \rightarrow UD$.

Appendix A

Tables of Numbers

For convenience of reference, we reproduce some tables of known values of M_n, m_n, $\overline{M}_n$, and $M_n^{(k)}$. Tables A.1, A.2 and A.3 are reproduced from [10] with the entry for m_{43} corrected to agree with M_{22} and [13]. Table A.4 is reproduced from [5].

n	M_n	n	M_n	n	M_n
1	1	9	933458	17	59923200729046
2	2	10	8152860	18	608188709574124
3	8	11	73424650	19	6234277838531806
4	42	12	678390116	20	64477712119584604
5	262	13	6405031050	21	672265814872772972
6	1828	14	61606881612	22	7060941974458061392
7	13820	15	602188541928	23	74661728661167809752
8	110954	16	5969806669034	24	794337831754564188184

Table A.1: The first 24 meandric numbers.

n	m_n	n	m_n	n	m_n
1	1	16	252939	31	5969806669034
2	1	17	933458	32	15012865733351
3	2	18	2172830	33	59923200729046
4	3	19	8152860	34	151622652413194
5	8	20	19304190	35	608188709574124
6	14	21	73424650	36	1547365078534578
7	42	22	176343390	37	6234277838531806
8	81	23	678390116	38	15939972379349178
9	262	24	1649008456	39	64477712119584604
10	538	25	6405031050	40	165597452660771610
11	1828	26	15730575554	41	672265814872772972
12	3926	27	61606881612	42	1733609081727968492
13	13820	28	152663683494	43	7060941974458061392
14	30694	29	602188541928		
15	110954	30	1503962954930		

Table A.2: The first 43 open meandric numbers.

n	$\overline{M}_n$	n	$\overline{M}_n$	n	$\overline{M}_n$
1	1	16	1053874	31	42126805350798
2	1	17	3328188	32	137494070309894
3	2	18	10274466	33	455792943581400
4	4	19	32786630	34	1493892615824866
5	10	20	102511418	35	4967158911871358
6	24	21	329903058	36	16341143303881194
7	66	22	1042277722	37	54480174340453578
8	174	23	3377919260	38	179830726231355326
9	504	24	10765024432	39	600994488311709056
10	1406	25	35095839848	40	1989761816656666392
11	4210	26	112670468128	41	6664356253639465480
12	12198	27	369192702554	42	22124273546267785420
13	37378	28	1192724674590	43	74248957195109578520
14	111278	29	3925446804750	44	247100408917982623532
15	346846	30	12750985286162	45	830776205506531894760

Table A.3: The first 45 semi-meandric numbers.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	8	42	262	1828	13820	110954	933458	8152860	73424650	678390116
2		2	12	84	640	5236	45164	406012	3772008	35994184	351173328	3490681428
3			5	56	580	5894	60312	624240	6540510	69323910	742518832	8028001566
4				14	240	3344	42840	529104	6413784	76980880	919032664	10941339452
5					42	990	17472	271240	3935238	54787208	742366152	9871243896
6						132	4004	85904	1569984	26200468	412348728	6230748192
7							429	16016	405552	8536890	161172704	2830421952
8								1430	63648	1860480	44346456	934582000
9									4862	251940	8356656	222516030
10										16796	994840	36936988
11											58786	3922512
12												208012

Table A.4: Meandric system number $M_n^{(k)}$ for small n and k.

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