A COMPREHENSIVE ANALYSIS
OF
POLYHEDRAL LIFT-AND-PROJECT METHODS

YU HIN AU AND LEVENT TUNCEL

Abstract. We consider lift-and-project methods for combinatorial optimization problems and focus mostly on those lift-and-project methods which generate polyhedral relaxations of the convex hull of integer solutions. We introduce many new variants of Sherali–Adams and Bienstock–Zuckerberg operators. These new operators fill the spectrum of polyhedral lift-and-project operators in a way which makes all of them more transparent, easier to relate to each other, and easier to analyze. We provide new techniques to analyze the worst-case performances as well as relative strengths of these operators in a unified way. In particular, using the new techniques and a recent result of Mathieu and Sinclair, we prove that the polyhedral Bienstock–Zuckerberg operator requires at least $\sqrt{2n} - \frac{3}{2}$ iterations to compute the matching polytope of the $(2n + 1)$-clique. We further prove that the operator requires approximately $\frac{n}{2}$ iterations to reach the stable set polytope of the $n$-clique, if we start with the fractional stable set polytope. Lastly, we show that some of the worst-case instances for the positive semidefinite Lovász–Schrijver lift-and-project operator are also bad instances for the strongest variants of the Sherali–Adams operator with positive semidefinite strengthenings, and discuss some consequences for integrality gaps of convex relaxations.

1. Introduction

Given a polytope $P \subseteq [0, 1]^n$, we are interested in its integer hull (i.e., the convex hull of 0,1 vectors in $P$), $P_I := \text{conv}(P \cap \{0,1\}^n)$. While it is impossible to efficiently find a description of $P_I$ for a general $P$ (unless $P = \mathbb{N}P$), we may use properties that we know are satisfied by points in $P_I$ to derive inequalities that are valid for $P_I$ but not $P$.

Lift-and-Project methods provide a systematic way to generate a sequence of convex relaxations of $P_I$, converging to the integer hull $P_I$. These methods go back to work by Balas and others in the late 1960s and the early 1970s. Some of the most attractive features of these methods are:

- Convex relaxations of $P_I$ obtained after $\mathcal{O}(1)$ iterations of the procedure are tractable provided $P$ is tractable. Here, tractable may mean either that the underlying linear optimization problem is polynomial-time solvable, say due to the existence of a polynomial-time weak separation oracle for $P$; or, more strongly, that $P$ has an explicitly given,
polynomial size representation by linear inequalities (we will distinguish between these two versions of tractability, starting with the strength chart given in Figure 1).

- Many of these methods use lifted (higher dimensional) representations for the relaxations. Such representations sometimes allow compact (polynomial size in the input) convex representations of exponentially many facets.

- Most of these methods allow easy addition of positive semidefiniteness constraints in the lifted space. This feature can make the relaxations much stronger in some cases, without sacrificing polynomial-time solvability (perhaps only approximately). Moreover, these semidefiniteness constraints can represent an uncountable family of defining linear inequalities, such as those of the theta body of a graph.

- Systematic generation of tighter and tighter relaxations converging to $P_I$ in at most $n$ rounds makes the strongest of these methods good candidates for utilization in generating polynomial-time approximation algorithms for hard problems, or for proving large integrality gaps (hence providing a negative result about approximability in the underlying hierarchy of relaxations).

In the last two decades, many lift-and-project operators have been proposed (see, for example, [SA90], [LS91], [BCC93], [Las01] and [BZ04]), and have been applied to various discrete optimization problems (see, for example, [SL96], [AKP02], [PVZ07] and [GL07]). Many families of facets of the stable set polytope of graphs are shown to be easily generated by these procedures [LS91, LT03]. Also studied are their performances on max-cut [Lau02], set covering [BZ04], set partitioning [SL96], TSP relaxations [CD01, Che05, CGGS13], and matching [ST99, ABN04, MS09]. For general properties of these operators and some comparisons among them, see [GT01], [Lau03] and [HT08].

![Diagram of lift-and-project operators](image)

**Figure 1.** A strength chart of lift-and-project operators.

Figure 1 provides a glimpse of the spectrum of polyhedral lift-and-project operators, as well as their semidefinite strengthened counterparts. The operators BCC (due to Balas, Ceria and Cornuéjols [BCC93]); LS$_0$, LS and LS$_+$ (due to Lovász and Schrijver [LS91]); SA (due to Sherali and Adams [SA90]); and BZ, BZ$_+$ (due to Bienstock and Zuckerberg [BZ04]) will be formally defined in the subsequent sections. The Las operator is due to Lasserre [Las01]. The boxed operators in the figure are the new ones proposed in the current paper, and each solid arrow in the chart denotes “is refined by” (i.e., the operator that is at the head of an arrow is stronger than that at the tail). For instance, when applied to the same set $P$, the LS$_0$ operator yields a
relaxation that is at least as tight as that obtained by applying the BCC operator. Since BCC admits a very short and elegant proof that it returns $P_I$ after $n$ iterations for every $P \subseteq [0,1]^n$, it follows immediately that every operator in Figure 1 converges to $P_I$ in at most $n$ iterations. Moreover, if one can prove an upper bound result for any operator $\Gamma$ in Figure 1, then the same result applies to all operators in the diagram that can be reached from $\Gamma$ by a directed path. Moreover, any lower bound result on the ZZ’ operator implies the same result for all polyhedral lift-and-project operators in Figure 1. Likewise, to obtain a lower bound result for all lift-and-project operators shown in the diagram, it suffices to show that the result holds for ZZ’ and LS. (For some bad instances for LS, see [Lau02] and [Che07].)

As seen in Figure 1, the strongest polyhedral lift-and-project operators known to date are LS, SA and ZZ. We are interested in these strongest operators because they provide the strongest tractable relaxations obtained this way. On the other hand, if we want to prove that some combinatorial optimization problem is difficult to attack by lift-and-project methods, then we would hope to establish them on the strongest existing hierarchy for the strongest negative results. For example, some of the non-approximability results on vertex cover are based on the LS operator [GMP+06, STT06], and some other integrality gap results are based on SA [CMM09].

Furthermore, it was shown in [CLRST13] that non-approximability results for the SA relaxations of approximate constraint satisfaction problems can be extended to lower bound results on the extension complexity (i.e., the smallest number of variables needed to represent a given set as the projection of a tractable set in higher dimension) of the max-cut and max 3-sat polytopes. (The reader may refer to [Yan91] for the first major progress on the extension complexity of polytopes that arise from combinatorial optimization problems, and [Goe09, FMP+12, Rot13] for some of the recent breakthroughs in this line of work.)

Therefore, by understanding the more powerful lift-and-project operators, we could either obtain better approximations for hard combinatorial optimization problems, or lay some of the groundwork for yet stronger non-approximability results. Moreover, we shall see that these analyses typically also lead to other crucial information about the underlying hierarchy of convex relaxations, such as their integrality gaps.

In this paper, we introduce many new variants of Sherali–Adams and Bienstock–Zuckerberg operators. These new operators fill the spectrum of polyhedral lift-and-project operators in a way which makes all of them more transparent, easier to relate to each other, and easier to analyze in a comprehensive way. We provide new techniques to analyze the worst-case performances as well as relative strengths of these operators in a unified way. Among other new operators, we construct two strong, semidefinite versions of the Sherali–Adams operator that we call SA and SA’+. There are other weaker versions of these operators in the recent literature called Sherali–Adams SDP which have been previously studied, among others, by Chlamtac and Singh [CS08] and Benabbas et al. [BGM10, BM10, BCGM11, BGMT12], even though our versions are the strongest yet. We show that, under certain conditions, the performances of SA’+ and ZZ’ (the ZZ operator enhanced with an additional positive semidefiniteness constraint) are closely related to each other. Since SA and SA’+ inherit many properties from the well-studied SA operator, our findings provide another venue to understanding and analyzing ZZ and ZZ’. As another by-product of our approach, we provide strengthened versions of the ZZ, ZZ’ operators. Most of our analyses apply to these stronger versions. Next, we utilize the tools we have established and prove that the ZZ operator requires at least $\sqrt{2n^2 - \frac{3}{2}}$ iterations to compute the matching polytope of the $(2n + 1)$-clique, and approximately $\frac{n^2}{2}$ iterations to compute the stable set polytope of the $n$-clique. This establishes the first examples in which ZZ requires more than $O(1)$ iterations to reach the integer hull. We also show that some well-known worst-case instances for LS and LS extend to give worst-case instances for SA and SA+. Finally, we conclude the paper by
illustrating how the analyses and the tools we provided may be used to prove integrality gaps for various classes of relaxations obtained from lift-and-project operators with some desirable invariance properties.

Several of our results can be seen as “approximate converses” of the refinement relationship among various lift-and-project operators. Such relationships are represented by dashed arrows in Figure 2. As we shall see, sometimes a weaker operator can be guaranteed to perform at least as well as a stronger one, by an appropriate increase of iterate number and/or certain assumptions on the given polytope $P$.

![Figure 2. An illustration of several restricted reverse refinement results in this paper.](image)

2. Preliminaries

In this section, we describe several lift-and-project operators that produce polyhedral relaxations, and establish some notation. One of the most fundamental ideas behind the lift-and-project approach is convexification, which can be traced back to Balas’ work on disjunctive cuts in the 1970s. For convenience, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$ herein. Observe that, given $P \subseteq [0, 1]^n$, if we have mutually disjoint sets $Q_1, \ldots, Q_\ell \subseteq P$ such that their union, $\bigcup_{i=1}^{\ell} Q_i$, contains all integral points in $P$, then we can deduce that $P_I$ is contained in $\text{conv}\left( \bigcup_{i=1}^{\ell} Q_i \right)$, which therefore is a potentially tighter relaxation of $P_I$ than $P$. Perhaps the simplest way to illustrate this idea is via the operator devised by Balas, Ceria and Cornuéjols [BCC93] which we call the BCC operator. Given $P \subseteq [0, 1]^n$ and an index $i \in [n]$, define

$$\text{BCC}_i(P) := \text{conv}\left( \{ x \in P : x_i \in \{0, 1\} \} \right).$$

Moreover, we can apply $\text{BCC}_i$ followed by $\text{BCC}_j$ to a polytope $P$ to make progress. In fact, it is well-known that for every $P \subseteq [0, 1]^n$,

$$\text{BCC}_1(\text{BCC}_2(\cdots(\text{BCC}_n(P))\cdots)) = P_I.$$ 

This establishes that for every polytope $P$, one can obtain its integer hull with at most $n$ applications of the BCC operator.

While iteratively applying BCC in all $n$ indices is intractable (unless $P = \mathcal{N}P$), applying them simultaneously to $P$ and intersecting them is not. Furthermore, it is easy to see that $P_I$
is contained in the intersection of these $n$ sets. Thus,

$$\text{LS}_0(P) := \bigcap_{i \in [n]} \text{BCC}_i(P),$$

devised by Lovász and Schrijver [LS91], is a relaxation of $P_I$ that is at least as tight as $\text{BCC}_i(P)$ for all $i \in [n]$. Figure 3 illustrates how BCC and $\text{LS}_0$ operate in two dimensions.

![Figure 3](image-url)

**Figure 3.** An illustration of BCC and $\text{LS}_0$ in two dimensions.

Before we look into operators that are even stronger (and more sophisticated), it is helpful to understand the following alternative description of $\text{LS}_0$. Given $x \in [0, 1]^n$, let $\hat{x}$ denote the vector $\left(1 \ x\right)$ in $\mathbb{R}^{n+1}$, where the new coordinate is indexed by 0. Let $e_i$ denote the $i$th unit vector (of appropriate size), and for any square matrix $M$, let $\text{diag}(M)$ denote the vector formed by the diagonal entries of $M$. Next, given $P \subseteq [0, 1]^n$, define the cone $K(P) := \left\{ \left( \begin{array}{c} \lambda \\ \lambda x \end{array} \right) \in \mathbb{R} \oplus \mathbb{R}^n : \lambda \geq 0, x \in P \right\}$. Then, it is not hard to check that

$$\text{LS}_0(P) = \left\{ x \in \mathbb{R}^n : \exists Y \in \mathbb{R}^{(n+1) \times (n+1)}, Ye_i, Ye_0 - ye_i \in K(P), \forall i \in [n], Ye_0 = Y^T e_0 = \text{diag}(Y) = \hat{x} \right\}.$$

In this perspective, perhaps the most straightforward way to see that $\text{LS}_0(P) \supseteq P_I$ is: given any integral vector $x \in P$, the matrix $Y := \hat{x} \hat{x}^T$ is a matrix which “certifies” that $x \in \text{LS}_0(P)$. Then $P_I \subseteq \text{LS}_0(P)$ follows from the fact that the latter is obviously a convex set.

Now, observe that $\hat{x} \hat{x}^T$ is symmetric for all $x \in \{0, 1\}^n$. Thus, if we let $S^n$ denote the set of $n$-by-$n$ real, symmetric matrices, then

$$\text{LS}(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in S^{n+1}, Ye_i, Ye_0 - ye_i \in K(P), \forall i \in [n], Ye_0 = \text{diag}(Y) = \hat{x} \right\}$$

also contains $P_I$. By enforcing a symmetry constraint on the matrices in the lifted space (and still retaining all integral points in $P$), we see that $\text{LS}(P)$ is a potentially tighter relaxation than $\text{LS}_0(P)$. We can also apply these operators iteratively to a polytope $P$ to gain progressively tighter relaxations. Let $\text{LS}_0^k(P)$ (resp. $\text{LS}^k(P)$) denote the set obtained from applying $\text{LS}_0$ (resp. $\text{LS}$) to $P$ iteratively for $k$ times. It is apparent from their definitions that $\text{LS}(P) \subseteq \text{LS}_0(P) \subseteq \text{BCC}_i(P)$, for every $i \in [n]$. Hence, it follows that $\text{LS}_0^k(P) = \text{LS}^k(P) = P_I$, for every $P \subseteq [0, 1]^n$.
to sets of even higher dimensions. From here on, we denote \( \{0,1\}^n \) by \( \mathcal{F} \), and \( \mathcal{A} := 2^\mathcal{F} \), the power set of \( \mathcal{F} \). For each \( x \in \mathcal{F} \), we define the vector \( x^\mathcal{A} \in \mathbb{R}^A \) where
\[
x_{\alpha}^\mathcal{A} = \begin{cases} 
1 & \text{if } x \in \alpha; \\
0 & \text{otherwise}. 
\end{cases}
\]
That is, each coordinate of \( \mathcal{A} \) corresponds to a subset of the vertices of the \( n \)-dimensional unit hypercube, and \( x_{\alpha}^\mathcal{A} = 1 \) if and only if the point \( x \) is contained in the set \( \alpha \). It is not hard to see that for all \( x \in \mathcal{F} \), we have \( x_{\emptyset}^\mathcal{F} = 1 \), and \( x_{\{y \in \mathcal{F} : y_i = 1\}}^\mathcal{F} = x_i, \forall i \in [n] \). Another important property of \( x^\mathcal{A} \) is that, given disjoint subsets \( \alpha_1, \alpha_2, \ldots, \alpha_k \subseteq \beta \subseteq \mathcal{F} \), we know that
\[
x_{\alpha_1}^\mathcal{A} + x_{\alpha_2}^\mathcal{A} + \cdots + x_{\alpha_k}^\mathcal{A} \leq x_{\beta}^\mathcal{A},
\]
and equality holds if \( \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \) partitions \( \beta \).

Thus, for any given \( x \in \mathcal{F} \), if we define \( Y_x^\mathcal{A} := x^\mathcal{A} (x^\mathcal{A})^\top \), then the entries of \( Y_x^\mathcal{A} \) have considerable structure. Most notably, the following must hold:

(P1) \( Y_x^\mathcal{A} e_\mathcal{F} = (Y_x^\mathcal{A})^\top e_\mathcal{F} = \text{diag}(Y_x^\mathcal{A}) = x^\mathcal{A} \);
(P2) \( Y_x^\mathcal{A} e_\alpha \in \{0, x^\mathcal{A}\}, \forall \alpha \in \mathcal{A} \);
(P3) \( Y_x^\mathcal{A} \in \mathbb{S}^\mathcal{A} \);
(P4) \( Y_x^\mathcal{A}[\alpha, \beta] = 1 \iff x \in \alpha \cap \beta \);
(P5) If \( \alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2 \), then \( Y_x^\mathcal{A}[\alpha_1, \beta_1] = Y_x^\mathcal{A}[\alpha_2, \beta_2] \);
(P6) Every row and column of \( Y_x^\mathcal{A} \) satisfies (1).

Of course, \( Y_x^\mathcal{A} \) has exponential size (in \( n \)), and explicitly constructing elements in a lifted space of such a high dimension could yield an intractable structure, which makes the underlying algorithm no better than simply enumerating the integral points in \( \mathcal{P} \). Nevertheless, we can try to obtain a tight relaxation by only working with polynomial size submatrices of \( Y_x^\mathcal{A} \), and imposing constraints that are relaxations of the above, in hope of capturing some important inequalities that are valid for \( \mathcal{P} \) but not \( P \). Zuckerberg \cite{Zuc03} showed that most of the existing lift-and-project operators can be interpreted under this common theme.

We next express the operators devised by Sherali and Adams \cite{SA90} in this language. Given a set of indices \( S \subseteq [n] \) and \( t \in \{0,1\} \), we define
\[
S|_t := \{x \in \mathcal{F} : x_i = t, \forall i \in S\}.
\]
To reduce cluttering, we write \( i|_t \) instead of \( \{i\}|_t \). Given any integer \( \ell \in \{0,1,\ldots,n\} \), we define \( \mathcal{A}_\ell := \{S|_1 \cap T|_0 : S, T \subseteq [n], S \cap T = \emptyset, |S| + |T| \leq \ell\} \) and \( \mathcal{A}_\ell^+ := \{S|_1 : S \subseteq [n], |S| \leq \ell\} \). Also, given any vector \( y \in \mathbb{R}^\mathcal{A} \) for some \( \mathcal{A}' \subseteq \mathcal{A} \) which contains \( \mathcal{F} \) and \( i|_1 \) for all \( i \in [n] \), we let \( \hat{x}(y) := (y_x, y_{i_1}, \ldots, y_{i_n})^\top \).

For any fixed integer \( k \in [n] \), the \( \mathsf{SA}^k \) operator can be defined as follows:

(1) Let \( \mathsf{SA}^k(P) \) denote the set of matrices \( Y \in \mathbb{R}^{\mathcal{A}_k^+ \times \mathcal{A}_k} \) which satisfy all of the following conditions:

(SA 1) \( Y[\mathcal{F}, \mathcal{F}] = 1 \).
(SA 2) \( \hat{x}(Ye_\alpha) \in K(P) \), for every \( \alpha \in \mathcal{A}_k \).
(SA 3) For every \( S|_1 \cap T|_0 \in \mathcal{A}_{k-1} \),
\[
Ye_{S|_1 \cap T|_0} + Ye_{S|_1 \cap T|_0} = Ye_{S|_1 \cap T|_0}, \quad \forall j \in [n] \setminus (S \cup T).
\]
(SA 4) For all \( \alpha \in \mathcal{A}_k^+, \beta \in \mathcal{A}_k \) such that \( \alpha \cap \beta = \emptyset \), \( Y[\alpha, \beta] = 0 \).
(SA 5) For all \( \alpha_1, \alpha_2 \in \mathcal{A}_k^+, \beta_1, \beta_2 \in \mathcal{A}_k \) such that \( \alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2 \), \( Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2] \).
(2) Define
\[
\mathsf{SA}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \mathsf{SA}^k(P), Ye_\mathcal{F} = \hat{x} \right\}.
\]
The SA$^k$ operator was originally described by linearizing polynomial inequalities, as follows: given an inequality \( \sum_{i=1}^{n} a_i x_i \leq a_0 \) that is valid for \( P \), disjoint subsets of indices \( S, T \subseteq [n] \) such that \( |S| + |T| \leq k \), SA$^k$ generates the inequality

\[
\left( \prod_{i \in S} x_i \right) \left( \prod_{i \in T} (1 - x_i) \right) \left( \sum_{i=1}^{n} a_i x_i \right) \leq \left( \prod_{i \in S} x_i \right) \left( \prod_{i \in T} (1 - x_i) \right) a_0,
\]

and obtains a linear inequality by replacing \( x_i^j \) with \( x_i \) (for all \( j \geq 2 \)) in all terms, and then by using a new variable to represent each nontrivial product of monomials. In our definition of SA$^k$, the linearized inequality would be

\[
\sum_{i=1}^{n} a_i Y[i, S]_1 \cap T_0 \leq a_0 Y[F, S_1 \cap T_0],
\]

which is enforced by (SA 2) on the column of \( Y \) indexed by the set \( S_1 \cap T_0 \). Also, for any set of indices \( U \subseteq [n] \), the product of monomials \( \prod_{i \in U} x_i \) could appear multiple times in the original formulation when we generate (2) using different \( S \) and \( T \). Then SA$^k$ identifies them all by the variable \( x_U \) in the linearized formulation. This requirement is enforced by (SA 5) in our definition.

It is not hard to see that SA$^1(P) = LS(P)$. In general, SA obtains extra strength over LS by lifting \( P \) to a set of matrices of higher dimension, and using some properties of sets in \( A \) to identify variables in the lifted space.

Finally, we look into the polyhedral lift-and-project operator devised by Bienstock and Zuckerberg [BZ04]. Recall that the idea of convexification requires a collection of disjoint subsets of \( P \) whose union contains all integral points in \( P \). So far, every operator that we have seen obtains these sets by intersecting \( P \) with some faces of \([0,1]^n\). However, sometimes it is beneficial to allow more flexibility in choosing the way we partition the integral points in \( P \). For example, consider

\[
P := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq n - \frac{1}{2} \right\}.
\]

In this case, SA$^{n-1}(P)$, a relaxation obtained from using convexification with exponentially many sets that are all intersections of \( P \) and faces of \([0,1]^n\), still strictly contains \( P_I \). On the other hand, if we define

\[
Q_j := \left\{ x \in P : \sum_{i=1}^{n} x_i = j \right\},
\]

for every \( j \in \{0,1,\ldots,n\} \), then every integral point in \( P \) is contained in \( Q_j \) for some \( j \), and

\[
P_I = \text{conv}\left( \bigcup_{i=0}^{n} Q_j \right).
\]

We will see in the next section that the set \( \text{conv}\left( \bigcup_{i=0}^{n} Q_j \right) \) can be described as the projection of a set of dimension \( O(n^2) \) that is tractable as long as \( P \) is.

Bienstock and Zuckerberg [BZ04] utilized this type of ideas and invented operators that use variables in \( A \) that were not exploited by the operators proposed earlier, in conjunction with some new constraints. We will denote their polyhedral operator by BZ, but we also present variants of it called BZ$'$ and BZ$''$. These modified operators have the advantage of being stronger, and are also simpler to present. Moreover, since we are mostly interested in applying these operators to polytopes that arise from set packing problems (such as the stable set and matching problems.
of graphs), we will state versions of these operators that only apply to lower-comprehensive polytopes. We will discuss this in more detail after stating the elements of their operators.

Suppose we are given a polytope \( P := \{ x \in [0,1]^n : Ax \leq b \} \), where \( A \in \mathbb{R}^{m \times n} \) is a nonnegative and \( b \in \mathbb{R}^m \) is positive. The BZ' operator can be viewed as a two-step process. The first step is refinement. Given a vector \( v \), let \( \text{supp}(v) \) denote the support of \( v \). Also, for every \( i \in [m] \), let \( A^i \) denote the \( i \)th row of \( A \). If \( O \subseteq [n] \) satisfies

- \( O \subseteq \text{supp}(A^i) \);
- \( \sum_{j \in O} A^i_j > b_i \); and
- \( |O| \leq k + 1 \) or \( |O| \geq |\text{supp}(A^i)| - (k + 1) \)

for some \( i \in [m] \), then we call \( O \) a \( k \)-small obstruction. Let \( \mathcal{O}_k \) denote the collection of all \( k \)-small obstructions of \( P \) (or more precisely, of the system \( Ax \leq b \)). Notice that, for every obstruction \( O \in \mathcal{O}_k \), and integral vector \( x \in P \), the inequality \( \sum_{i \in O} x_i \leq |O| - 1 \) holds. Thus,

\[
\mathcal{O}_k(P) := \left\{ x \in P : \sum_{i \in O} x_i \leq |O| - 1, \ \forall O \in \mathcal{O}_k \right\}
\]

is a relaxation of \( P_I \) that is potentially tighter than \( P \).

The second step of the BZ\(^k\) operator is lifting. Before we give the details of this step, we need another intermediate set of indices, called walls. For every \( k \geq 1 \), we define

\[
\mathcal{W}_k := \left\{ \bigcup_{i,j \in [\ell], i \neq j} (O_i \cap O_j) : O_1, \ldots, O_\ell \in \mathcal{O}_k, \ell \leq k + 1 \right\} \cup \{\{1\}, \ldots, \{n\}\}.
\]

That is, each subset of up to \( (k + 1) \) \( k \)-small obstructions generate a wall, which is the set of elements that appear in at least two of the given obstructions. We also ensure that the singleton sets of indices are walls. Next, we define the collection of tiers

\[
\mathcal{T}_k := \left\{ S \subseteq [n] : \exists W_{i_1}, \ldots, W_{i_k} \in \mathcal{W}_k, S \subseteq \bigcup_{j=1}^k W_{i_j} \right\}.
\]

That is, we define a set of indices \( S \) to be a tier if there exist \( k \) walls whose union contains \( S \). Note that every subset of \([n]\) of size up to \( k \) is a tier. Finally, given a set \( U \subseteq [n] \) and a nonnegative integer \( r \), we define

\[
|U|_{<r} := \left\{ x \in F : \sum_{i \in U} x_i \leq r - 1 \right\}.
\]

We shall see that the elements in \( \mathcal{A} \) that are being generated by BZ' all take the form \( S \cap T \cap U |_{<r} \), where \( S, T, U \) are disjoint sets of indices. Next, we describe the lifting step of BZ\(^k\):

1. **Define \( \mathcal{A}' \) to be the set consisting of the following.** For each tier \( S \in \mathcal{T}_k \), include:

   - \( (S \setminus T)|_1 \cap T|_0 \),
   - for all \( T \subseteq S \) such that \( |T| \leq k \);

   - \( (S \setminus (T \cup U))|_1 \cap T|_0 \cap U |_{<|U|-(k-|T|)} \),
   - for every \( T, U \subseteq S \) such that \( U \cap T = \emptyset, |T| < k \) and \( |U| + |T| > k \).

   We say these variables (indexed by the above sets) are associated with the tier \( S \).

2. **Let \( \tilde{BZ}^k(P) \) denote the set of matrices \( Y \in \mathbb{S}^{\mathcal{A}'} \) that satisfy all of the following conditions:**

\[
\begin{align*}
&\text{(1) Define } \mathcal{A}' \text{ to be the set consisting of the following. For each tier } S \in \mathcal{T}_k, \text{ include:} \nonumber \\
&\text{• } (S \setminus T)|_1 \cap T|_0, \\
&\text{for all } T \subseteq S \text{ such that } |T| \leq k; \\
&\text{• } (S \setminus (T \cup U))|_1 \cap T|_0 \cap U |_{<|U|-(k-|T|)}, \\
&\text{for every } T, U \subseteq S \text{ such that } U \cap T = \emptyset, |T| < k \text{ and } |U| + |T| > k. \\
\end{align*}
\]
Thus, $A$ refines $SA$. Notice that in $BZ$, the variables in $BZ$, which we have intentionally suppressed in $BZ'$ only polynomially many are selected. The role of walls is also much more important in selecting operators and the original Bienstock–Zuckerberg operators are given in the appendix.

and analysis more transparent. Some of the details of the relationships between these modified $k$ indices, $BZ$ also generates exponentially many variables, whereas in the original $BZ$ only polynomially many are selected. The role of walls is also much more important in selecting the variables in $BZ$, which we have intentionally suppressed in $BZ'$ to make our presentation and analysis more transparent. Some of the details of the relationships between these modified operators and the original Bienstock–Zuckerberg operators are given in the appendix.


(BZ' 2) For every column $x$ of the matrix $Y$,

(i) $0 \leq x_{\alpha} \leq x_F$, for all $\alpha \in A'$.

(ii) $\hat{x}(x) \in K(O_k(P))$.

(iii) $x_{i 1} + x_{i 0} = x_F$, for every $i \in [n]$.

(iv) For each $\alpha \in A'$ in the form $S|1 \cap T|0$ impose the inequalities

\[ x_{i 1} \geq x_{\alpha}, \quad \forall i \in S; \]

\[ x_{i 0} \geq x_{\alpha}, \quad \forall i \in T; \]

\[ x_\alpha + x_{(S \cup \{i\})|1 \cap (T \setminus \{i\})|0} = x_{S|1 \cap (T \setminus \{i\})|0}, \quad \forall i \in T; \]

\[ \sum_{i \in S} x_{i 1} + \sum_{i \in T} x_{i 0} - x_{\alpha} \leq (|S| + |T| - 1)x_F. \]

(v) For each $\alpha \in A'$ in the form $S|1 \cap T|0 \cap U| < r$, impose the inequalities

\[ x_{i 1} \geq x_{\alpha}, \quad \forall i \in S; \]

\[ x_{i 0} \geq x_{\alpha}, \quad \forall i \in T; \]

\[ \sum_{i \in U} x_{i 0} \geq (|U| - (r - 1))x_{\alpha}; \]

\[ x_{\alpha} = x_{S|1 \cap T|0} - \sum_{U \subseteq U, |U| \geq r} x_{(S \cup U')|1 \cap (T \cup (U \setminus U'))|0}. \]

(BZ' 3) For all $\alpha, \beta \in A'$ such that $\alpha \cap \beta \cap P = \emptyset$, $Y[\alpha, \beta] = 0$.

(BZ' 4) For all $\alpha_1, \beta_1, \alpha_2, \beta_2 \in A'$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

(3) Define $BZ^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in BZ^k(P), \hat{x}(Ye_F) = \hat{x} \right\}$.

Similar to the case of $SA^k$, $BZ^k$ can be seen as creating columns that correspond to sets that partition $F$. While $SA^k$ only generates a partition for each subset of up to $k$ indices, $BZ^k$ does so for every tier, which is a much broader collection of indices. For a tier $S$ up to size $k$, it does the same as $SA^k$ and generates $2^{|S|}$ columns corresponding to all possible negations of indices in $S$. However, for $S$ of size greater than $k$, it generates a “$k$-deep” partition of $S$: a column for $(S \setminus T)|1 \cap T|0$ for each $T \subseteq S$ of size up to $k$, and a column for $S|_< |S|_k$. In fact, given a tier $S$ and $T \subseteq S$ such that $|T| < k$, $BZ^k$ generates a $(k - |T|)$-deep partition of this set for each $U \subseteq S \setminus T$ such that $|U| + |T| > k$. First, the column for $$(S \setminus (T \cup U'))|1 \cap (T \cup U')|0 = (S \setminus (T \cup U'))|1 \cap T|0 \cap (U \setminus U')|1 \cap U'|0 \cap U|_< |(k - |T|)$$ is generated for all $U' \subseteq U$ of size $\leq k - |T|$. Then $BZ^k$ also generates $$(S \setminus (T \cup U'))|1 \cap T|0 \cap U|_< |U| -(k - |T|)$$ to capture the remainder of the partition.

Since each singleton index set is a wall, we see that every index set of size up to $k$ is a tier. Thus, $A'$ contains $A_k$, and it is not hard to see that $BZ^k(P) \subseteq SA^k(O_k(P))$ in general. $(BZ^k$ also refines $SA^k$, a stronger version of $SA^k$ that will be defined after the next theorem.) Furthermore, notice that in $BZ'$, we have generated exponentially many variables, whereas in the original $BZ$ only polynomially many are selected. The role of walls is also much more important in selecting the variables in $BZ$, which we have intentionally suppressed in $BZ'$ to make our presentation and analysis more transparent. Some of the details of the relationships between these modified operators and the original Bienstock–Zuckerberg operators are given in the appendix.
One of the main results Bienstock and Zuckerberg achieved with the BZ$^k$ operator is on set covering problems. Given an inequality $a^\top x \geq a_0$ such that $a \geq 0$ and $a_0 > 0$, its pitch is defined to be the smallest positive integer $j$ such that

$$S \subseteq \text{supp}(a), |S| \geq j \Rightarrow a^\top \chi^S \geq a_0.$$ 

Also, let $\bar{e}$ denote the all-ones vector of suitable size. Then they showed the following powerful result:

**Theorem 1** (Bienstock and Zuckerberg [BZ04]). Suppose $P := \{x \in [0,1]^n : Ax \geq \bar{e}\}$ where $A$ is a $0,1$ matrix. Then for every $k \geq 1$, every valid inequality of $P_1$ that has pitch at most $k+1$ is valid for BZ$^k(P)$.

Note that if all coefficients of an inequality are integral and at most $k$, then the pitch of the inequality is no more than $k$. An important property of the Bienstock–Zuckerberg operators is that its performance can vary upon different algebraic descriptions of the given set $P$, even if they geometrically describe the same set. For instance, adding a redundant inequality to the lifted space which correspond to integral points in $P$ whenever $Ax \leq b$ could make many more sets qualify as $k$-small obstructions. This could increase the dimension of the lifted set as more walls and tiers are generated, and as a result strengthen the operator. We provide examples that illustrate this phenomenon in the appendix.

Next, we take a closer look into the condition (BZ$^4$ 3), which is one of the conditions used in the Bienstock–Zuckerberg operators that were not explicitly imposed by the earlier lift-and-project operators. Observe that, for any $x \in P \cap \{0,1\}^n$,

$$Y^x_\alpha[\alpha,\beta] = x^\top A_\alpha \cdot A_\beta = 0$$

whenever $\alpha \cap \beta \cap P = \emptyset$. Thus, imposing such a constraint still preserves all matrices in the lifted space which correspond to integral points in $P$.

Since we will relate the performance of BZ$'$ and BZ$'_4$ to other operators (such as SA), it is worthwhile to investigate how this new condition impacts the performance of an operator. Given $P \subseteq [0,1]^n$, and integer $k \geq 1$, define

$$\text{SA}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\text{SA}}^k(P) : Ye_F = \hat{x} \right\}.$$ 

where $\tilde{\text{SA}}^k(P)$ is the set of matrices in $\tilde{\text{SA}}^k(P)$ that satisfy

(SA$'$ 4) For all $\alpha \in A^+_1, \beta \in A_k$ such that $\alpha \cap \beta \cap P = \emptyset$, $Y[\alpha,\beta] = 0$.

Note that $\text{SA}^k$ yields a tractable algorithm when $k = O(1)$, as the condition (SA$'$ 4) can be verified efficiently (assuming $P$ is tractable), and is only checked polynomially many times. Also, since (SA$'$ 4) is more restrictive than (SA 4), it is apparent that $\text{SA}^k(P) \subseteq \text{SA}^k(P)$ in general. However, it turns out that in the case of SA, this extra condition would at most “save” one iteration.

**Proposition 2.** For every $P \subseteq [0,1]^n$ and every $k \geq 1$,

$$\text{SA}^{k+1}(P) \subseteq \text{SA}^k(P).$$

**Proof.** Let $x \in \text{SA}^{k+1}(P)$, and let $Y \in \tilde{\text{SA}}^{k+1}(P)$ such that $Ye_F = \hat{x}$. Define $Y' \in \mathbb{R}^{A^+_1 \times A_k}$ such that $Y'[\alpha,\beta] = Y[\alpha,\beta], \forall \alpha \in A^+_1, \beta \in A_k$ (i.e., $Y'$ is a submatrix of $Y$). Since $Y'e_F = Ye_F = \hat{x}$, it suffices to show that $Y' \in \tilde{\text{SA}}^k(P)$.

By construction, it is obvious that $Y' \in \tilde{\text{SA}}^k(P)$. Thus, we just need to show that $Y'$ satisfies (SA$'$ 4). Given $\alpha \in A^+_1, \beta \in A_k$, suppose $\alpha = i|_1$, and $\beta = S|_1 \cap T|_0$ for $S,T \subseteq \{n\}$. Now $\alpha \cap \beta = (S \cup \{i\})|_1 \cap T|_0 \in A_{k+1}$, and thus the entry $Y[F,\alpha \cap \beta]$ exists.
Since \( Y_{\alpha \cap \beta} \in K(P) \) by (SA 2), \( Y[F, \alpha \cap \beta] > 0 \) would imply that the point
\[
\frac{1}{Y[F, \alpha \cap \beta]}(Y[1|1, \alpha \cap \beta], Y[2|1, \alpha \cap \beta], \ldots, Y[n|1, \alpha \cap \beta])^T
\]
is in \( P \), and thus \( \alpha \cap \beta \cap P \neq \emptyset \). Hence, we see that (SA’ 4) holds, and our claim follows. \( \square \)

Proposition 2 establishes the dashed arrow from SA’ to SA in Figure 2 and assures that if one can provide a performance guarantee for SA’ on a polytope \( P \), then the same can be said of the weaker SA operator by using one extra iteration. The meanings for the other four dashed arrows in Figure 2 are similar in nature — for some linear or quadratic function of the iterate number, the weaker operator can be at least as strong as the stronger operator. However, they are much more involved than Proposition 2 and sometimes depend on the properties of the given set \( P \). We will address them in detail in the subsequent sections.

3. Identifying Unhelpful Variables in the Lifted Space

As we have seen in the previous section, one way to gain additional strength in devising a lift-and-project operator is to lift to a space of higher dimension, and obtain a potentially tighter formulation by using more variables, albeit at a computational cost. In this section, we provide conditions on sets and higher dimensional liftings which do not lead to strong cuts. As a result, we show in some cases, BZ\(^k \) performs no better than SA\(^\ell \) for some suitably chosen \( k \) and \( \ell \).

3.1. A General Template. Recall that \( \mathcal{F} = \{0, 1\}^n \), and \( \mathcal{A} \) is the power set of \( \mathcal{F} \). A common theme among all lift-and-project operators we have looked at so far is that their lifted spaces can all be interpreted as sets of matrices whose columns and rows are indexed by elements in \( \mathcal{A} \). Moreover, they all impose a constraint in the tune of “each column of the matrix belongs to a certain set linked to \( P^n \) (e.g. conditions (SA 2) and (BZ’ 2)). This provides a natural way of partitioning the constraints of a lift-and-project operator into two categories: those that are present (and identical) for every matrix column, and the remaining constraints that cannot be captured this way.

We say that a lift-and-project operator \( \Gamma \) is admissible if it possesses all of the following properties:

(I1) Given a convex set \( P \subseteq [0, 1]^n \), \( \Gamma \) lifts \( P \) to a set of matrices \( \hat{\Gamma}(P) \subseteq \mathbb{R}^{S \times S'} \), such that
\[
\mathcal{A}^+_1 \subseteq S \subseteq S' \subseteq \mathcal{A}.
\]

(I2) There exists a column constraint function \( f \) that maps elements in \( \mathcal{A} \) to subsets of \( \mathbb{R}^S \), and a cross-column constraint function \( g \) that maps sets contained in \( [0, 1]^n \) to sets of matrices in \( \mathbb{R}^{S \times S'} \), such that
\[
\hat{\Gamma}(P) = \{ Y \in g(P) : Ye_{S'} \in f(S'), \forall S' \in S' \}.
\]

Furthermore, \( f \) has the property that, for every pair of disjoint sets \( S, T \in S' \):

1. \( f(S) \cup f(T) \subseteq f(S \cup T) \);
2. \( f(S) = f(T) \) if \( S \cap P = T \cap P \).

(I3)
\[
\Gamma(P) := \{ x \in \mathbb{R}^n : \exists Y \in \hat{\Gamma}(P), Y[F, \mathcal{F}] = 1, \hat{x}(Ye_F) = \hat{x} \}.
\]

Note that the notion of admissible operators is extremely broad. For any lift-and-project operator \( \Gamma \), we can show that it is admissible by letting \( g(P) := \hat{\Gamma}(P) \) and \( f(S) := \mathbb{R}^S \) for all \( S \in \mathcal{A} \) (i.e., we define \( f \) to be trivial and “shove” all constraints of \( \Gamma \) under \( g \)). However, as mentioned above, the intention of this definition is that we try to capture as much of \( \Gamma \) as possible with \( f \) by using it to describe the constraints \( \Gamma \) places on every column of the matrices.
in the lifted space, and only include the remaining constraints in $g$. Thus, we want $f$ to be maximal, and $g$ to be minimal in this sense. For instance, we can show that $\text{SA}^k$ is admissible by defining $f(S) := K(P \cap \text{conv}(S))$, $\forall S \in \mathcal{A}$ and $g(P)$ to be the set of matrices in $\mathbb{R}^{A_j^i \times A_k}$ that satisfy (SA 3), (SA 4) and (SA 5). All named operators mentioned in this manuscript can be shown to be admissible in this fashion — using $f$ to describe that each matrix column has to be in some lifted set determined by $P$, and let $g$ capture the remaining constraints.

For many known operators, these “other” constraints placed by $g$ are relaxations of the set theoretical properties (P5) and (P6) of $Y^A_x$. For instance, (SA 5) is in place to make sure the variables in the linearized polynomial inequalities that would be identified in the original description of $\text{SA}^k$ would in fact have the same value in all matrices in $\tilde{\text{SA}}^k(P)$. Likewise, (SA 3) and (SA 4) are also needed to capture the relationship between the variables that would be established naturally in the original description with polynomial inequalities.

Furthermore, sometimes using matrices to describe the lifted space and assigning set theoretical meanings to their columns and rows have advantages over using linearized polynomial inequalities directly. For instance, we again consider the set

$$P := \left\{ x \in [0,1]^n : \sum_{i=1}^n x_i \leq n - \frac{1}{2} \right\}.$$ 

We have seen that if we define

$$Q_j := \left\{ x \in P : \sum_{i=1}^n x_i = j \right\},$$

for every $j \in \{0,1,\ldots,n\}$, then $P_I = \text{conv} \left( \bigcup_{j=0}^n Q_j \right)$. However, if we attempt to construct a formulation by linearizing polynomial inequalities as in the original description of $\text{SA}$, then to capture the constraints for $Q_j$ one would need to linearize

$$\sum_{S:T:S,T=[n]} \left( \prod_{i \in S} x_i \right) \left( \prod_{i \in T} (1 - x_i) \right) \left( \sum_{i=1}^n a_i x_i \right) \leq \sum_{S:T:S,T=[n]} \left( \prod_{i \in S} x_i \right) \left( \prod_{i \in T} (1 - x_i) \right) a_0$$

for all inequalities $\sum_{i=1}^n a_i x_i \leq a_0$ that are valid for $P$. Of course, when $j \approx \frac{n}{2}$, the above constraint would have exponentially many terms.

However, we can obtain an efficient lifted formulation by doing the following: for each $j \in \{0,1,\ldots,n\}$, define $R_j \in \mathcal{A}$ where

$$R_j = \left\{ x \in \mathbb{F} : \sum_{i=1}^n x_i = j \right\},$$

and let $S = \{ \mathcal{F}, R_0, R_1, \ldots, R_n \}$. We now define $\Gamma$ to be the lift-and-project operator as follows:

1. Let $\hat{\Gamma}(P)$ denote the set of matrices $Y \in \mathbb{R}^{A_j^i \times S}$ such that
   (i) $Y[\mathcal{F}, \mathcal{F}] = 1$,
   (ii) $Ye_{R_j} \in K(P \cap \text{conv}(R_j))$, $\forall j \in \{0,\ldots,n\}$.
   (iii) $Ye_{\mathcal{F}} = \sum_{j=0}^n Ye_{R_j}$.
2. Define
   $$\Gamma(P) = \left\{ x \in \mathbb{R}^n : \exists Y \in \hat{\Gamma}(P), Ye_{\mathcal{F}} = x \right\}.$$
Then it is not hard to see that \( \Gamma(P) = \text{conv}(\bigcup_{i=0}^{n} Q_j) \) for any set \( P \subseteq [0,1]^n \). Note that we used constraint (iii) to enforce that the entries in the matrix behave consistently with their corresponding set theoretical meanings. In particular, since \( R_0, \ldots, R_n \) partition \( F \), we require that the columns indexed by the sets \( R_0, \ldots, R_n \) sum up to that representing \( F \).

Thus, the following notions are helpful when we attempt to analyze \( g \) more systematically. First, given \( S, S' \subseteq A \), we say that \( S' \) refines \( S \) if for all \( S \in S \), there exists mutually disjoint sets in \( S' \) that partition \( S \). For example, \( A_k \) refines \( A_\ell \) whenever \( k \geq \ell \). Note that the notion of refinement is transitive — if \( S \) refines \( S' \) and \( S' \) refines \( S'' \), then \( S \) refines \( S'' \).

Next, given \( Y_1 \in \mathbb{R}^{S_1 \times S'_1} \) and \( Y_2 \in \mathbb{R}^{S_2 \times S'_2} \) where \( S_1, S'_1, S_2, S'_2 \subseteq A \), we say that \( Y_1 \) and \( Y_2 \) are consistent if
\[
\sum_{i=1}^{k} Y_1[S_{1i}, S'_{1i}] = \sum_{i=1}^{\ell} Y_2[S_{2i}, S'_{2i}]
\]
whenever \( \{S_{1i} \cap S'_{1i} : i \in [k]\} \) and \( \{S_{2i} \cap S'_{2i} : i \in [\ell]\} \) are collections of mutually disjoint sets such that \( \bigcup_{i=1}^{k} (S_{1i} \cap S'_{1i}) = \bigcup_{i=1}^{\ell} (S_{2i} \cap S'_{2i}) \). Note that we will also use the above definition to define whether two vectors are consistent with each other, whether a matrix and a vector are consistent with each other.

We say that a matrix \( Y \in \mathbb{R}^{S \times S'} \), where \( S, S' \subseteq A \), is overall measure consistent (OMC) if it is consistent with itself. All matrices in the lifted spaces of \( \text{SA}^k, \text{SA}^\ell \) and \( \text{BZ}^k \) satisfy (OMC), for all \( k \geq 1 \). One notable observation is that, in the case of vectors, if \( S' \) refines \( S \), and \( x \in \mathbb{R}^{S'} \) satisfies (OMC), then there is a unique \( y \in \mathbb{R}^S \) that is consistent with \( x \).

Finally, we are ready to formally describe some variables that we will show are unhelpful in the lifted space under this framework. Given an admissible operator \( \Gamma \) and \( P \subseteq [0,1]^n \), suppose \( \hat{\Gamma}(P) \subseteq \mathbb{R}^{S \times S'} \). We say that \( S' \) is \( P \)-useless if there is a collection \( T = \{T_1, \ldots, T_k\} \subseteq S' \) such that

1. \( S' \subseteq T \) and \( \bigcup_{i=1}^{k} T_i \subseteq S' \setminus T \);
2. there exists \( \ell \in [k] \) such that \( P \cap \text{conv}(T_j) = \emptyset, \forall j \neq \ell \).

What does it mean for variables to be \( P \)-useless? For example, let \( T = \{T_1, \ldots, T_k\} \subseteq S' \) be a collection of variables such that \( R := \bigcup_{i=1}^{k} T_i \) is itself a variable in \( S' \), and \( R \notin T \). Further suppose there exists a unique \( \ell \in [k] \) such that \( P \cap \text{conv}(T_j) = \emptyset, \forall j \neq \ell \). This means that in this setting, each of the \( T_j \) has the set theoretical meaning of the empty set. Thus, we do not lose any points when projecting \( \hat{\Gamma}(P) \) to \( \Gamma(P) \) if we assume that the matrix column indexed by \( T_j \) is the vector of all zeros. Moreover, since \( R = \bigcup_{i=1}^{k} T_i \), the variables \( R \) and \( T_\ell \) can be interpreted as having the same set theoretical meaning in the formulation, and we can deem \( T_\ell \) redundant. Therefore, in this case, \( T_1, \ldots, T_k \) are all \( P \)-useless.

With the notion of \( P \)-useless variables, we can show the following:

**Proposition 3.** Let \( \Gamma_1, \Gamma_2 \) be two admissible lift-and-project operators, \( P \subseteq [0,1]^n \), and suppose \( \hat{\Gamma}_1(P) \in \mathbb{R}^{S_1 \times S'_1} \) and \( \hat{\Gamma}_2(P) \in \mathbb{R}^{S_2 \times S'_2} \). Also, let \( f_1, g_1 \) and \( f_2, g_2 \) be the corresponding constraint functions of \( \Gamma_1 \) and \( \Gamma_2 \) respectively, and let \( U \) be a set of \( P \)-useless variables in \( S'_2 \). Further suppose that the following conditions hold:

1. Every matrix in \( \hat{\Gamma}_1(P) \) satisfies (OMC).
2. \( \{S \cap S' : S \in S_1, S' \in S'_1\} \) refines \( \{S \cap S' : S \in S_2 \setminus U, S' \in S'_2 \setminus U\} \), and \( S'_1 \) refines \( S'_2 \setminus U \).
3. Let \( Y \in \hat{\Gamma}_1(P) \), and \( S \in S'_2 \). If \( y \in \mathbb{R}^{S_2 \times \{S\}} \) is consistent with \( Y \), then \( y \in f_2(S) \).
4. If \( Y_1 \in g_1(P) \) and \( Y_2 \in \mathbb{R}^{S_2 \times S'_2} \) is consistent with \( Y_1 \), then \( Y_2 \in g_2(P) \).

Then, \( \Gamma_1(P) \subseteq \Gamma_2(P) \).
Intuitively, the above conditions are needed so that given a point \( x \in \Gamma_1(P) \) and its certificate matrix \( Y \in \tilde{\Gamma}(P) \), we know enough structure about the entries and set theoretic meanings of \( Y \) to construct a matrix in \( \mathbb{R}^{(S_2 \setminus U) \times (S_2' \setminus U)} \) that is consistent with \( Y \). Then using the fact that the variables in \( U \) are \( P \)-useless, we can extend this to a matrix in \( \mathbb{R}^{S_2 \times S_2'} \) that certifies \( x \)'s membership in \( \Gamma_2(P) \). Also, for \( y \in \mathbb{R}^{S_2 \times \{S\}} \), we are referring to a vector with \(|S_2|\) entries that are indexed by elements of \( \{T \cap S : T \in S_2\} \). Since we will be talking about whether \( y \) is consistent with another vector or matrix, we will need to specify not only the entries of \( y \), but also these entries’ corresponding sets.

Now we are ready to prove Proposition \[3\]

**Proof of Proposition \[3\]** Suppose \( x \in \Gamma_1(P) \). Let \( Y \in \mathbb{R}^{S_1 \times S_1'} \) be a matrix in \( \tilde{\Gamma}(P) \) such that \( \hat{x}(Ye_F) = \hat{x} \). First, we construct an intermediate matrix \( Y' \in \mathbb{R}^{(S_2 \setminus U) \times (S_2' \setminus U)} \). For each \( \alpha \in S_2 \setminus U \) and \( \beta \in S_2' \setminus U \), we know (due to (\(ii\))) that there exists a set of ordered pairs

\[
I_{\alpha,\beta} \subseteq \{(S, S') : S \in S_1, S' \in S_1'\}
\]

such that the collection \( \{S \cap S' : (S, S') \in I_{\alpha,\beta}\} \) partitions \( \alpha \cap \beta \). Next, we construct \( Y' \) such that

\[
Y'[\alpha, \beta] := \sum_{(S, S') \in I_{\alpha,\beta}} Y[S,S'].
\]

Note that by (OMC), the entry \( Y'[\alpha, \beta] \) is invariant under the choice of \( I_{\alpha,\beta} \). Also, since \( \{(F, F)\} \) is a valid candidate for \( I_{F,F} \), we see that \( Y'[F,F] = Y[F,F] = 1 \), and \( \hat{x}(Y'e_F) = \hat{x}(Ye_F) = \hat{x}. \)

Next, we construct \( Y'' \in \tilde{\Gamma}_2(P) \) from \( Y' \). Given \( \alpha \in U \) such that \( P \cap \text{conv}(\alpha) \neq \emptyset \), there exists a set \( h(\alpha) \in S_2' \setminus U \) such that \( \text{conv}(\alpha) \cap P = \text{conv}(h(\alpha) \cap P) \). Note that \( h(\alpha) \) may not be unique, but any eligible choice would do.

Next, we define \( V^1 \in \mathbb{R}^{(S_2 \setminus U) \times S_2'} \) as follows:

\[
V^1(e_{\alpha}) := \begin{cases} 
  e_{\alpha} & \text{if } \alpha \in S_2 \setminus U; \\
  e_{h(\alpha)} & \text{if } \alpha \in U \text{ and } \text{conv}(\alpha) \cap P \neq \emptyset; \\
  0 & \text{otherwise.}
\end{cases}
\]

Similarly, we define \( V^2 \in \mathbb{R}^{(S_2' \setminus U) \times S_2'} \) as follows:

\[
V^2(e_{\alpha}) := \begin{cases} 
  e_{\alpha} & \text{if } \alpha \in S_2' \setminus U; \\
  e_{h(\alpha)} & \text{if } \alpha \in U \text{ and } \text{conv}(\alpha) \cap P \neq \emptyset; \\
  0 & \text{otherwise.}
\end{cases}
\]

We show that \( Y'' := V^1Y'(V^2) \dagger \in \tilde{\Gamma}_2(P) \). Since our map from \( Y \) to \( Y'' \) preserves (OMC), \( Y'' \) is consistent with \( Y \), and thus by (iv) it satisfies all constraints in \( g_2 \). Also, by (iii) it satisfies all column constraints in \( f_2 \) as well. Thus, \( Y'' \in \Gamma_2(P) \). Since \( \hat{x}(Y''e_F) = \hat{x} \), we are finished. \( \square \)

We note that, in some cases, we can relate the performance of two lift-and-project operators by assuming a condition slightly weaker than (OMC). Given a matrix \( Y \in \mathbb{R}^{S \times S'} \), where \( S, S' \subseteq A \), we say that it is row and column measure consistent (RCMC) if every column and row of \( Y \) satisfies (OMC). As is apparent in its definition, (RCMC) is less restrictive than (OMC). In fact, it is satisfied by all matrices in the lifted space of all named lift-and-project operators mentioned in this paper. Then, we have the following result that is the (RCMC) counterpart of Proposition \[3\]

**Proposition 4.** Let \( \Gamma_1, \Gamma_2 \) be two admissible lift-and-project operators, \( P \subseteq [0, 1]^n \), and suppose \( \Gamma_1(P) \in \mathbb{R}^{S_1 \times S_1'} \) and \( \Gamma_2(P) \in \mathbb{R}^{S_2 \times S_2'} \). Also, let \( f_1, g_1 \) and \( f_2, g_2 \) be the corresponding constraint
functions of \( \Gamma_1 \) and \( \Gamma_2 \) respectively, and let \( U \) be a set of \( P \)-useless variables in \( S'_2 \). Further suppose that all of the following conditions hold:

(i) Every matrix in \( \tilde{\Gamma}_1(P) \) satisfies (RCMC).
(ii) \( S_1 \) refines \( S_2 \setminus U \), and \( S'_1 \) refines \( S'_2 \setminus U \).
(iii) Let \( S \in S'_2 \). If \( x \in \mathbb{R}^{S_1 \times \{S\}} \) is contained in \( f_1(S) \) and \( y \in \mathbb{R}^{S_2 \times \{S\}} \) is consistent with \( x \), then \( y \in f_2(S) \).
(iv) If \( Y_1 \in g_1(P) \) and \( Y_2 \in \mathbb{R}^{S_2 \times S'_2} \) is consistent with \( Y_1 \), then \( Y_2 \in g_2(P) \).

Then, \( \Gamma_1(P) \subseteq \Gamma_2(P) \).

Proof. The result can be shown by following the same outline as in the proof of Proposition 3. Suppose \( x \in \Gamma_1(P) \) and \( Y \in \mathbb{R}^{S_1 \times S'_1} \) is a certificate matrix for \( x \). For each \( \alpha \in S_2 \setminus U \), define \( I_\alpha \) to be a collection of sets in \( S_1 \) that partitions \( \alpha \). Since \( S_1 \) refines \( S_2 \setminus U \), such a collection must exist. Likewise, for all \( \alpha \in S'_2 \setminus U \), we define \( I'_\alpha \) to be a collection of sets in \( S'_1 \) that partitions \( \alpha \).

Next, we define \( Y' \in \mathbb{R}^{(S_2 \setminus U) \times (S'_2 \setminus U)} \) such that

\[
Y'_{\alpha, \beta} := \sum_{S \in I_\alpha, S' \in I'_\beta} Y[S, S'].
\]

Since \( Y \) satisfies (RCMC), \( Y'_{\alpha, \beta} \) is invariant under the choices of \( I_\alpha \) and \( I'_\beta \). From here on, we can define \( V_1, V_2 \) and \( Y'' \in \mathbb{R}^{S_2 \times S'_2} \) as in the proof of Proposition 3, and apply the same reasoning therein to show that it is in \( \Gamma_2(P) \). Now since \( \hat{x}(Y''e_F) = \hat{x}(Ye_F) = \hat{x} \), we conclude that \( x \in \Gamma_2(P) \).

\[\Box\]

3.2. Implications and Applications. Next, we look into several applications of Proposition 3 and 4. First, it is apparent that given two operators \( \Gamma_1, \Gamma_2 \) and set \( P \) such that \( \Gamma_1(P) \subseteq \Gamma_2(P) \), the integrality gap of \( \Gamma_1(P) \) is no more than that of \( \Gamma_2(P) \) with respect to any chosen direction. We will formally define integrality gap and discuss these results in more depth in Section 5.

Next, we relate the performance of \( BZ' \) and \( SA' \) under some suitable conditions. First, we define a tier \( S \in T_k \) to be \( P \)-useless if all variables associated with \( S \) are \( P \)-useless. Then we have the following:

**Theorem 5.** Suppose there exists \( \ell \in [n] \) such that all tiers \( S \) generated by \( BZ^k \) of size greater than \( \ell \) are \( P \)-useless. Then

\[
BZ^k(P) \supseteq \text{SA}^{2\ell}(O_k(P)).
\]

Proof. Let \( \Gamma_1 = \text{SA}^{2\ell}(O_k(\cdot)) \) and \( \Gamma_2 = BZ^k(\cdot) \). We prove our assertion by checking all conditions listed in Proposition 3.

First of all, all matrices in the lifted space of \( \text{SA}^{2\ell} \) satisfy (OMC). Next, since \( S'_1 = A_1^\ell \) and \( S'_2 = A_2^\ell \), we see that \( \{S \cap S' : S \in S_1, S' \in S'_1\} \) refines \( A_2^\ell \). On the other hand, since every tier of size greater than \( \ell \) is \( P \)-useless, we see that \( A_\ell \) refines both \( S_2 \setminus U \) and \( S'_2 \setminus U \). Thus, \( A_2^\ell \) refines \( \{S \cap S' : S \in S_2 \setminus U, S' \in S'_2 \setminus U\} \). Also, it is apparent that \( S'_1 = A_2^\ell \) refines \( S'_2 \setminus U \), so (ii) holds.

For (iii), we let \( f_1(S) = K(O_k(P) \cap \text{conv}(S)), \forall S \in A \), and

\[
f_2(S) := \left\{ y \in \mathbb{R}^{S'_2} : \hat{x}(y) \in K(O_k(P) \cap \text{conv}(S)), y \text{ satisfies } (BZ'2) \right\}.
\]

Note that all conditions in (BZ'2) are relaxations of constraints in (P5) and (P6), and thus are implied by (OMC). Let \( Y \in \text{SA}^{2\ell}(O_k(P)) \), and \( Y'' \) be the matrix obtained from the construction in the proof of Proposition 3. Since \( Y \) satisfies (OMC) and \( Y'' \) is consistent with \( Y \), the columns of \( Y'' \) must satisfy (BZ'2).
To check (iv), we see that $g_2(P)$ would be the set of matrices in the lifted space that satisfy (BZ’ 3) and (BZ’ 4). It is easy to see that (BZ’ 4) is implied by (OMC). For (BZ’ 3), suppose $S \in \mathcal{S}_2, S' \in \mathcal{S}_2$, and $S \cap S' \cap \mathcal{O}_k(P) = \emptyset$. If $Y''[S, S'] \neq 0$, then we know that $P \cap \text{conv}(S) \neq \emptyset$ and $P \cap \text{conv}(S') \neq \emptyset$, by the construction of $Y''$. Thus, define $\alpha := S$ if $S \not\subseteq U$, and $\alpha := h(S)$ if $S \subseteq U$. Likewise, define $\beta := S'$ if $S' \not\subseteq U$, and $\beta := h(S')$ if $S' \subseteq U$. In all cases, we have now obtained $\alpha \in \mathcal{S}_2 \setminus U, \beta \in \mathcal{S}_2 \setminus U$ such that $Y''[\alpha, \beta] = Y''[S, S']$.

Since

$$Y''[\alpha, \beta] = Y''[\alpha, \beta] = \sum_{(T,T') \in \mathcal{I}_{\alpha,\beta}} Y[T,T'],$$

we obtain $T \in \mathcal{A}_1^+, T' \in \mathcal{A}_k$ such that $Y[T,T'] \neq 0$. Then by (SA’ 4), $T \cap T' \cap P \neq \emptyset$. This implies that $S \cap S' \cap P \neq \emptyset$, and so (BZ’ 3) holds. \qed

We remark that, with a little more care and using the same observation as in the proof of Proposition 2, one can slightly sharpen Theorem 5 and show that SA$^{\mathcal{O}_k(P)} \subseteq \text{BZ}_k(P)$ under these assumptions.

Next, we look into the lift-and-project ranks some graph-based polytopes. For any lift-and-project operator $\Gamma$ and polytope $P$, we define the $\Gamma$-rank of $P$ to be the smallest integer $k$ such that $\Gamma^k(P) = P_1$. The notion of rank gives us a measure of how close $P$ is to $P_1$ with respect to $\Gamma$. Moreover, it is useful when comparing the performances of different operators.

Given a simple, undirected graph $G = (V,E)$, we define

$$MT(G) := \left\{ x \in [0,1]^E : \sum_{j : (i,j) \in E} x_{ij} \leq 1, \forall i \in V \right\}.$$ 

Then $MT(G)_1$ is the matching polytope of $G$, and is exactly the convex hull of the incidence vectors of the matchings of $G$.

While there exist efficient algorithms that solve the matching problem (e.g., Edmonds’ seminal blossom algorithm [Edm65]), many lift-and-project operators have been shown to require exponential time to compute the matching polytope starting with $MT(G)$. In particular, $MT(K_{2n+1})$ is known to have LS$_+$-rank $n$ [ST99] and BCC-rank $n^2$ [ABN04]. More recently, Mathieu and Sinclair [MS09] showed that the SA-rank of $MT(K_{2n+1})$ is $2n - 1$. Using their result and Theorem 5, we can show that this polytope is also a bad instance for BZ$'$.

Theorem 6. Let $G$ be the complete graph on $2n + 1$ vertices. Then the BZ$'$-rank of $MT(G)$ is at least $\left\lceil \sqrt{2n - \frac{3}{2}} \right\rceil$.

Proof. Let $G = K_{2n+1}$ and $P = MT(G)$. We first identify the tiers generated by BZ$^k$ that are $P$-useless. Observe that a set $O \subseteq E$ is a $k$-small obstruction generated by BZ$^k$ if there is a vertex that is incident with all edges in $O$, and that $|O| \leq k + 1$ or $|O| \geq 2n - k$. Now suppose $W \in \mathcal{W}_k$ is a wall, and let $\{e_1, e_2, \ldots, e_p\}$ be a maximum matching contained in $W$. Notice that for $e_1 = \{u_1, v_1\}$ to be in $W$, it has to be contained in at least 2 obstructions, and each of these obstructions has to originate from the $u_1$- or $v_1$-constraint in the formulation of $MT(G)$. Now suppose $e_2 = \{u_2, v_2\}$. By the same logic, we deduce that the obstructions that allow $e_2$ to be in $W$ have to be different from those that enabled $e_1$ to be in $W$. Since each wall is generated by at most $k + 1$ obstructions, we see that $p \leq \frac{k+1}{2}$. Therefore, for every tier $S \in \mathcal{T}_k$ (which has to be contained in the union of $k$ walls), the maximum matching contained in $S$ has at most $\frac{k(k+1)}{2}$ edges.

Hence, if $|S| > \frac{k(k+1)}{2} + k$, then $S \setminus T$ is not a matching for any set $T \subseteq S$ of size up to $k$, which implies $(S \setminus T)_{|1} \cap T_{|0} \cap P = \emptyset$. Thus, the only variables $\alpha$ associated with $S$ such that
Proposition 2 and the Mathieu–Sinclair result, we see that the \(\alpha\) variable (see that \(\alpha\) strengthens BZ). Thus, by Theorem 5, for BZ' best upper bound to roughly and Sinclair’s result, and the fact that BZ' is defined to be implied by \(k \geq \sqrt{2n - \frac{3}{2}}\).

The best upper bound we know for the BZ'-rank of \(MT(K_{2n+1})\) is \(2n - 1\) (due to Mathieu and Sinclair's result, and the fact that BZ' refines SA^k). We shall see in the next section that strengthening BZ' by an additional positive semidefiniteness constraint decreases the current best upper bound to roughly \(\sqrt{2n}\).

We next look at the stable set problem of graphs. Given a graph \(G = (V, E)\), its fractional stable set polytope is defined to be

\[
FRAC(G) := \{ x \in [0, 1]^V : x_i + x_j \leq 1, \forall \{i, j\} \in E \}.
\]

Then the stable set polytope \(STAB(G) := FRAC(G)_I\) is precisely the convex hull of incidence vectors of stable sets of \(G\). Since there is a bijection between the set of matchings in \(G\) and the set of stable sets in its line graph \(L(G)\), we have the next result readily from Theorem 6.

**Corollary 7.** Let \(G\) be the line graph of \(K_{2n+1}\). Then the BZ'-rank of \(FRAC(G)\) is at least \(\lceil \sqrt{2n - \frac{3}{2}} \rceil\).

**Proof.** First, it is not hard to see that \(MT(H) \subseteq FRAC(L(H))\), for every graph \(H\). Also, it is apparent from the definition of BZ' that \(O_k(P) \subseteq O_k(P')\) implies \(BZ^k(P) \subseteq BZ^k(P')\). Since the collection of \(k\)-small obstructions of \(FRAC(G)\) is exactly the set of edges of \(G\) for all \(k \geq 1\), we see that \(FRAC(G) = O_k(FRAC(G))\). Therefore,

\[
O_k(MT(K_{2n+1})) \subseteq MT(K_{2n+1}) \subseteq FRAC(G) = O_k(FRAC(G)),
\]

which implies that the BZ'-rank of \(FRAC(G)\) is at least that of \(MT(K_{2n+1})\).

Thus, we obtain from Corollary 7 a family of graphs on \(n\) vertices whose fractional stable set polytope has BZ'-rank \(\Omega(n^{1/4})\).

We next turn to the complete graph \(G := K_n\). It is well-known that \(FRAC(G)\) has rank \(\Theta(n)\) with respect to SA (and as a result, all weaker operators such as LS and LS_0). We show that this is also true for BZ'.

**Theorem 8.** Suppose \(G\) is the complete graph on \(n \geq 3\) vertices. Then the BZ'-rank of \(FRAC(G)\) is between \(\lceil \frac{n}{2} \rceil - 2\) and \(\lceil \frac{n+1}{2} \rceil\). The same bounds apply for the BZ-rank.

The proof of Theorem 8 will be provided in the appendix. Thus, we see that, like all other popular polyhedral lift-and-project operators, BZ' (which is already stronger than BZ) performs poorly on the fractional stable set polytope of complete graphs.


Up to this point, we have looked exclusively at lift-and-project operators that produce polyhedral relaxations, where the main tool operators use to gain strength is to lift a given relaxation to a higher dimensional space. In this section, we turn our focus to operators that do not produce polyhedral relaxations. In particular, we will introduce several lift-and-project operators that utilize positive semidefiniteness, and look into the power and limitations of these additional constraints.
4.1. Lift-and-Project Operators with Positive Semidefiniteness. Perhaps the most elementary operator of this type is the \( \text{LS}_+ \) operator defined in \cite{LS91}. Recall that one way to see why \( P \subseteq \text{LS}(P) \) in general is to observe that for any integral point \( x \in P, \hat{x}x^\top \) is a matrix that certifies \( x \)'s membership in \( \text{LS}(P) \). Since \( \hat{x}x^\top \) is positive semidefinite for all \( x \), if we let \( \mathbb{S}_+^n \subseteq \mathbb{S}^n \) denote the set of symmetric, positive semidefinite \( n \)-by-\( n \) matrices, then it is easy to see that

\[ \text{LS}_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \mathbb{S}_+^{n+1}, Y_{e_i}, Y(e_0 - e_i) \in K(P), \forall i \in [n], Y_{e_0} = \text{diag}(Y) = \hat{x} \right\} \]

contains \( P \) as well. Also, by definition, \( \text{LS}_+(P) \subseteq \text{LS}(P) \) for all \( P \subseteq [0,1]^n \), and thus \( \text{LS}_+ \) potentially obtains a tighter relaxation than \( \text{LS}(P) \) in general.

Likewise, we can also define positive semidefinite variants of \( \text{SA} \). Given any positive integer \( k \), we define the operators \( \text{SA}^k_+ \) and \( \text{SA}^k_+ \) as follows:

1. Let \( \tilde{\text{SA}}^k_+(P) \) denote the set of matrices \( Y \in \mathbb{S}_+^{A_k} \) that satisfy all of the following conditions: (SA\(_1\)) \((SA_1)\), (SA\(_2\)) \((SA_2)\) and (SA\(_3\)) \((SA_3)\).
2. Let \( \tilde{\text{SA}}^k_+(P) \) denote the set of matrices \( Y \in \mathbb{S}_+^{A_k} \) that satisfy (SA\(_1\)), (SA\(_3\)) and:
3. Define
   \[ \text{SA}^k_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\text{SA}}^k_+(P), \hat{x}(Y_{e_x}) = \hat{x} \right\}, \]
   and
   \[ \text{SA}^k_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\text{SA}}^k_+(P), \hat{x}(Y_{e_x}) = \hat{x} \right\}. \]

The \( \text{SA}^k_+ \) and \( \text{SA}^k_+ \) operators extend the lifted space of the \( \text{SA}_k \) operator to a set of square matrices, and impose an additional positive semidefiniteness constraint. What sets these two new operators apart is that \( \text{SA}^k_+ \) utilizes a \((BZ_3)\)-like constraint to potentially obtain additional strength over \( \text{SA}^k_+ \). While we have seen in their polyhedral counterparts \( \text{SA}' \) and \( \text{SA} \) that adding this additional constraint could decrease the rank of a polytope by at most one, we shall provide an example later in this section in which the \( \text{SA}'_+ \)-rank of a polytope is lower than the \( \text{SA}_+ \)-rank by \( \Theta(n) \).

Next, we show that \( \text{SA}^k_+ \) refines the \( \text{LS}_+ \) operator (i.e., \( k \) iterative applications of \( \text{LS}_+ \)):

**Proposition 9.** For every polytope \( P \subseteq [0,1]^n \) and every integer \( k \geq 1 \),

\[ \text{SA}^k_+(P) \subseteq \text{LS}_+(\text{SA}^{k-1}_+(P)). \]

**Proof.** Suppose \( Y \in \tilde{\text{SA}}^k_+(P) \) and \( \hat{x}(Y_{e_x}) = \hat{x} \). Let \( Y' \) be the \((n+1)\)-by-\((n+1)\) symmetric minor of \( Y \), with rows and columns indexed by elements in \( A_k^+ \). To adapt to the notation for \( \text{LS}_+ \), we index the rows and columns of \( Y' \) by \( 0, 1, \ldots, n \) (instead of \( F, 1, e_1, \ldots, n, e_n \)). It is obvious that \( Y' \in \mathbb{S}^{n+1}_+ \), and \( Y'e_0 = \text{diag}(Y') = \hat{x} \). Thus, it suffices to show that \( Y'e_1, Y'(e_0 - e_i) \in K(\text{SA}^{k-1}_+(P)), \forall i \in [n] \).

We first show that \( Y'e_1 \in K(\text{SA}^{k-1}_+(P)) \). If \( (Y'e_1)_0 = 0 \), then \( Y'e_1 \) is the zero vector and the claim is obviously true. Next, suppose \( (Y'e_1)_0 > 0 \). Define the matrix \( Y'' \in \mathbb{S}^{A_k-1} \), such that

\[ Y''[\alpha, \beta] = \frac{1}{(Y'e_1)_0} Y[\alpha \cap 1_1, \beta \cap 1_1], \quad \forall \alpha, \beta \in A_{k-1}. \]

Notice that \( Y'' \) is a positive scalar multiple of a symmetric minor of \( Y \), and thus is positive semidefinite. Moreover, it satisfies (SA\(_1\)) by construction, and inherits the properties (SA\(_2\)), (SA\(_3\)), (SA\(_2\)) and (SA\(_3\)) from \( Y \). Thus, \( Y'' \in \tilde{\text{SA}}^{k-1}_+(P) \) and \( \hat{x}(Y''_{e_x}) = \frac{1}{(Y'e_1)_0} Y'e_1 \in \text{LS}_+(\text{SA}^{k-1}_+(P)). \)
than \( \ell \). By symmetry, it follows that \( Y' e_i \in K(SA_{+}^{k-1}(P)) \), \( \forall i \in [n] \). The argument for \( Y'(e_0 - e_i) \) is analogous.

It follows immediately from Proposition \( \Box \) that \( SA_{+}^{k}(P) \subseteq LS_{+}^{k}(P) \). The \( SA_{+}^{r} \) and \( SA'_{+} \) operators will be useful in simplifying our analysis and improving our understanding of the Bienstock–Zuckerberg operator enhanced with positive semidefiniteness, which is defined as

\[
BZ_{+}^{k}(P) := \left\{ x \in \mathbb{R}^{n} : \exists Y \in \tilde{BZ}_{+}^{k}(P), \hat{x}(Ye_{\mathcal{F}}) = \hat{x} \right\},
\]

where \( \tilde{BZ}_{+}^{k}(P) := BZ_{+}^{k}(P) \cap S_{+}^{k} \).

### 4.2. Unhelpful variables in PSD relaxations

We see that in Proposition 4, in the special case of comparing two lift-and-project operators whose lifted spaces are both square matrices (i.e. \( S_{1} = S_{1}' \) and \( S_{2} = S_{2}' \)), the construction of \( Y' \) and \( Y'' \) preserves positive semidefiniteness of \( Y \). Thus, this framework can be applied even when \( g_{1} \) and \( g_{2} \) enforce positive semidefiniteness constraints in their respective lifted spaces. The following is an illustration of such an application:

**Theorem 10.** Suppose there exists \( \ell \in [n] \) such that all tiers \( S \) generated by \( BZ_{+}^{k} \) of size greater than \( \ell \) are \( P \)-useless. Then

\[
BZ_{+}^{k}(P) \supseteq SA_{+}^{\ell}(O_{k}(P)).
\]

**Proof.** We prove our claim by verifying the conditions in Proposition 4. First, every matrix in the lifted space of \( SA_{+}^{\ell} \) satisfies (OMC), which implies (RCMC). Next, since \( S_{1} = S_{1}' = A_{\ell} \) and every tier of \( BZ_{+}^{k} \) that is not useless has size at most \( \ell \), we see that (ii) holds as well.

For (iii), note that we can let

\[
f_{1}(S) = \left\{ y \in \mathbb{R}^{S_{1}} : \hat{x}(y) \in K(P \cap \text{conv}(S)), y \text{ satisfies (OMC)} \right\},
\]

and

\[
f_{2}(S) = \left\{ y \in \mathbb{R}^{S_{2}} : \hat{x}(y) \in K(P \cap \text{conv}(S)), y \text{ satisfies (BZ' 2)} \right\}.
\]

As mentioned before, all conditions in (BZ' 2) are implied by (OMC) constraints and the fact that \( A_{\ell} \) refines \( S_{2} \). Thus, (iii) is satisfied.

For (iv), we see that \( g_{2}(P) \) would be the set of matrices in \( S_{+}^{\ell} \) that satisfy (BZ' 3) and (BZ' 4). It is easy to see that (BZ' 4) is implied by (OMC). Also, (BZ' 3) is implied by (SA'_{+} 2). Thus, we are finished. \( \Box \)

### 4.3. Utilizing \( \ell \)-establishing variables

Somewhat complementary to the notion of useless variables, here we look into instances where the presence of a certain set of variables in the lifted space provides a guarantee on the overall performance of the operator. Given \( j \in \{0, 1, \ldots, n\} \), let \( [n]_{j} \) denote the collection of subsets of \( [n] \) of size \( j \). Suppose \( Y \in S_{A'}^{\ell} \) for some \( A' \subseteq A \). We say that \( Y \) is \( \ell \)-established if all of the following conditions hold:

\[
\]

\( (\ell 2) \; Y \succeq 0. \)

\( (\ell 3) \; A_{\ell}^{+} \subseteq A'. \)

\( (\ell 4) \; \text{For all} \; \alpha, \beta, \alpha', \beta' \in A_{\ell}^{+} \text{ such that} \; \alpha \cap \beta = \alpha' \cap \beta', \; Y[\alpha, \beta] = Y[\alpha', \beta']. \)

\( (\ell 5) \; \text{For all} \; \alpha, \beta \in A_{\ell}^{+}, \; Y[F, \beta] \succeq Y[\alpha, \beta]. \)

Notice that all matrices in \( SA_{+}^{\ell}(P) \) (which contains \( SA_{+}^{\ell}(P) \)) are \( \ell \)-established, for all \( P \subseteq [0, 1]^{n} \). A matrix in \( \tilde{BZ}_{+}^{k}(P) \) is also \( \ell \)-established if all subsets of size up to \( \ell \) are generated as tiers. Given such a matrix, we may define \( \mathcal{M} := \bigcup_{i=0}^{2\ell} [n]_{i} \) and \( y \in \mathbb{R}^{\mathcal{M}} \) such that \( y_{S} = Y[S', 1, S''|1] \), where \( S', S'' \) are subsets of \( [n] \) of size at most \( \ell \) such that \( S' \cup S'' = S \). Note that such choices
of $S', S''$ must exist by (ℓ3), and by (ℓ4) the value of $y_S$ is invariant under the choices of $S'$ and $S''$.

Finally, we define $Z \in \mathbb{R}^{2\ell+1}$ such that

$$Z_i := \sum_{S \subseteq [n]_i} y_S, \quad \forall i \in \{0, 1, \ldots, 2\ell\}.$$ 

Note that $Z_0$ is always equal to 1 (by (ℓ1)), and $Z_1 = \sum_{i=1}^{n} Y[i|1, F]$. Also, observe that the entries of $Z$ are related to each other. For example, if $\hat{x}(Y_{e,F})$ is an integral 0-1 vector, then by (ℓ5) we know that $y_S \leq 1$ for all $S$, and $y_S > 0$ only if $y_{\{i\}} = 1, \forall i \in S$. Thus, we can infer that

$$Z_j = \sum_{S \in [n]_j} y_S \leq \binom{Z_1}{j}, \quad \forall j \in [2\ell].$$

We next show that the positive semidefiniteness of $Y$ also forces the $Z_i$’s to relate to each other, somewhat similarly to the above. The following result would be more intuitive by noting that $(\binom{p}{i+1}) / (\binom{p}{i}) = \frac{p-i}{i+1}$.

**Proposition 11.** Suppose $Y \in S^A_+$ is $\ell$-established, and $y, Z$ are defined as above. If there exists an integer $p \geq \ell$ such that

$$Z_{i+1} \leq \binom{p-i}{i+1} Z_i, \quad \forall i \in \{\ell, \ell+1, \ldots, 2\ell-1\},$$

then $Z_i \leq \binom{p}{i}$, $\forall i \in [2\ell]$. In particular, $Z_1 \leq p$.

**Proof.** We first show that $Z_\ell \leq \binom{p}{\ell}$. Given $i \in [\ell]$, define the vector $v(i) \in \mathbb{R}^A$ such that

$$v(i)_\alpha := \begin{cases} \binom{p}{\ell} & \text{if } \alpha = F; \\ -1 & \text{if } \alpha = [n]_i \text{ where } S \in [n]_i; \\ 0 & \text{otherwise.} \end{cases}$$

By the positive semidefiniteness of $Y$, we obtain

$$0 \leq v(\ell)^\top Y v(\ell) = \binom{p}{\ell}^2 - 2 \binom{p}{\ell} Z_\ell + \sum_{S, S' \in [n]_\ell} Y[S|1, S'|1].$$

Notice that for any $T \in [n]_{\ell+j}$, the number of sets $T', T'' \in [n]_\ell$ such that $T' \cup T'' = T$ is $\binom{j}{\ell} \binom{\ell+j}{\ell}$. Hence, this is the number of times the term $y_T$ appears in $\sum_{S, S' \in [n]_\ell} Y[S|1, S'|1]$. We also know by assumption that for all $j \in [\ell]$,

$$Z_{\ell+j} \leq \binom{p-j-\ell+1}{j+\ell} \binom{p-j-\ell+2}{j+\ell-1} \cdots \binom{p-\ell}{\ell+1} Z_\ell = \left(\frac{(p-\ell)! \ell!}{(p-\ell-j)! (j+\ell)!}\right) Z_\ell.$$
Since $Z$, and we deduce that $Z$. Suppose $\forall Z$.

Proof. $\forall \exists$.

Hence, $\forall \exists$.

Therefore, we conclude from $\exists$ that $0 \leq \binom{p}{2} - \binom{p}{i} Z_\ell$, which implies that $Z_\ell \leq \binom{p}{i}$. Together with $\exists$, this implies that $Z_{\ell+1} \leq \binom{p}{i}$, $\forall j \in \{0, 1, \ldots, \ell\}$.

It remains to show that $Z_i \leq \binom{p}{i}$, $\forall i \in [i - 1]$. To do that, it suffices to show that $Z_i \leq \binom{p}{i}$ can be deduced from assuming $Z_{i+j} \leq \binom{p}{i+j}$, $\forall j \in [i]$. Then applying the argument recursively would yield the result for all $i$. Observe that

$$\sum_{S,S' \in [n]} Y[S|1, S'|1] = \sum_{j=0}^{\ell} \sum_{S \in [n]_{\ell+j}} \left(\begin{array}{c} \ell+j \\ j \end{array}\right) y_S$$

$$= \sum_{j=0}^{\ell} \left(\begin{array}{c} \ell+j \\ j \end{array}\right) Z_{\ell+j}$$

$$\leq \sum_{j=0}^{\ell} \left(\begin{array}{c} \ell+j \\ j \end{array}\right) \left(\frac{(\ell+j)!}{j!}\right) \left(\frac{(p-\ell)!}{(\ell-j)!(\ell+j)!}\right) Z_\ell$$

$$= \binom{p}{\ell} Z_\ell.$$

Hence,

$$0 \leq v(i)^\top Y v(i) \leq \binom{p}{i}^2 - 2 \binom{p}{i} Z_i + \left(\binom{p}{i} - \binom{p}{i}^2\right) = 2 \left(\binom{p}{i} - 1\right) \left(\binom{p}{i} - Z_i\right),$$

and we conclude that $Z_i \leq \binom{p}{i}$. \hfill $\square$

An immediate but noteworthy implication of Proposition $\exists$ is the following:

**Corollary 12.** Suppose $Y \in \mathcal{S}^A$ is $\ell$-established, and $y, Z$ are defined as before. If $Z_i = 0$, $\forall i > \ell$, then $Z_1 \leq \ell$.

**Proof.** Since $Y \succeq 0$,

$$0 \leq v(\ell)^\top Y v(\ell) = 1 - 2Z_\ell + \sum_{S,S' \in [n]_\ell} Y[S|1, S'|1].$$

Since $Z_i = 0$, $\forall i > \ell, Y[S|1, S'|1] > 0$ only if $S = S'$. Therefore,

$$\sum_{S,S' \in [n]_\ell} Y[S|1, S'|1] = \sum_{S \in [n]_\ell} Y[S|1, S|1] = Z_\ell,$$

and we deduce that $Z_\ell \leq 1$. Then we can apply Proposition $\exists$ and deduce that $Z_i \leq \binom{p}{i}$, $\forall i \in [2\ell]$. In particular, $Z_1 \leq \ell$. \hfill $\square$
We now employ the upper bound proving techniques earlier and the notion of $\ell$-established matrices to prove the following result on the matching polytope of graphs.

**Theorem 13.** The $\text{SA}_+^\ell$-rank of $MT(K_{2n+1})$ is at most $n - \left\lfloor \frac{\sqrt{2n+1}-1}{2} \right\rfloor$.

**Proof.** Suppose $G = K_{2n+1}$ and let $P = MT(G)$. Let $Y \in \text{SA}_+^n(P)$. Since $Y$ is $k$-established, it suffices to show that $Z_{i+1} \leq \left( \frac{n+i}{i+1} \right) Z_i$ for all integer $i \in \{k,k+1,\ldots,2k-1\}$ whenever $k \geq n - \left\lfloor \frac{\sqrt{2n+1}-1}{2} \right\rfloor$. Then it follows from Proposition 11 that $Z_1 \leq n$, which implies $\sum_{i=1}^n x_i \leq n$ is valid for $\text{SA}_+^k(P)$.

By the fact that the maximum matching in $G$ has size $n$ and the condition $(\text{SA}_+^\ell, 2)$, $Z_i = 0$, $\forall i > n$. Thus, it suffices to verify the above claim for the case when $k \leq i \leq n-1$. Let $S$ be a matching of size $k$ that saturates the vertices $\{2n-2k+2,\ldots,2n+1\}$, let $T$ be a matching of size $i-k$ that saturates vertices $\{2n-2i+2,\ldots,2n-2k+1\}$ and let $E'$ be the set of edges that do not saturate any vertices in $S$ or $T$. Also, for each $U \subseteq E'$, we define the vector $f_U \in \mathbb{R}^{|E'|+1}$ (indexed by $\{0\} \cup E'$) such that

$$
(f_U)_i := \begin{cases} Y[(T \cup U)|_1 \cap (E' \setminus U)|_0, S|_1] & \text{if } i = 0 \text{ or if } i \in U; \\ 0 & \text{otherwise.} \end{cases}
$$

Notice that $k \geq n - \frac{\sqrt{2n+1}-1}{2}$ implies $k \geq \frac{(2n+1-2k)}{2} \geq |E'| + |T|$. Therefore, the above entries in $Y$ do exist, and the vectors $f_U$ are well defined. Also, applying (SA 3) iteratively on each index in $E'$ gives

$$
(13) \quad \sum_{|U| \leq |E'|} f_U = (Y[T|_1, S|_1], Y[(T \cup \{e_1\})|_1, S|_1], \ldots, Y[(T \cup \{e_{|E'|}\})|_1, S|_1])^T,
$$

where $e_1, \ldots, e_{|E'|}$ are the edges in $E'$.

Moreover, observe that $f_U = \begin{pmatrix} (f_U)_0 \\ (f_U)_0 \chi_U \end{pmatrix}$ for all $U \subseteq E'$, and by (SA^\ell_+) we know that $(f_U)_0 > 0$ only if $U \cup T \cup S$ is a matching of $G$, which implies that $U$ is a matching contained in $E'$. Since $E'$ spans $2n-2i+1$ vertices, such a $U$ must have size at most $n-i$. Thus, for each $f_U$ such that $(f_U)_0 > 0$, we know that $\sum_{i \in E'} (f_U)_i \leq (n-i)(f_U)_0$. Therefore, by [13],

$$
\left( \frac{2n-2i+1}{2} \right) \frac{Z_{i+1}}{|\mathcal{M}_{i+1}|} = \sum_{i \in E'} Y[(T \cup \{e_i\})|_1, S|_1] \leq (n-i)Y[T|_1, S|_1] = (n-i) \frac{Z_i}{|\mathcal{M}_i|}.
$$

Hence, $Z_{i+1} \leq \frac{n-1}{i+1} Z_i$. This concludes the proof, as it is easy to see that the facets of $MT(G)$ corresponding to smaller odd cliques in $G$ are also generated by $\text{SA}_+^k$.

Recall that the LS$_+^\ell$-rank of $MT(K_{2n+1})$ is exactly $n$. Thus, the techniques we proposed prove that $\text{SA}_+^\ell$ performs strictly better on this family of polytopes.

Next, we show that the notion of $\ell$-established matrices can also be applied to provide an upper bound on the BZ$_+^\ell$-rank of $MT(K_{2n+1})$.

**Theorem 14.** Suppose $G = K_{2n+1}$. Then the BZ$_+^\ell$-rank of $MT(G)$ is at most $\left\lceil \sqrt{2n+1} + \frac{1}{4} - \frac{1}{2} \right\rceil$.

**Proof.** Let $P = MT(K_{2n+1})$. First, we show that every subset $W \subseteq E$ of size up to $\left\lfloor \frac{k+1}{2} \right\rfloor$ is a wall generated by BZ$_+^k$. Given any edge $\{i, j\} \in W$, take a vertex $v \notin \{i, j\}$. Then $\{\{i, v\}, \{i, j\}\}$ and $\{\{j, v\}, \{i, j\}\}$ are both $k$-small obstructions for any $k \geq 1$, and their intersection contains
Proof. We prove our claim by constructing a matrix in $I \subseteq \mathbb{R}^A_k \times A_k$ that certifies $x \in \text{SA}_+^k(P)$. Recall that $A_k = \{S|_1 \cap T|_0 : S, T \subseteq [n], S \cap T = \emptyset, |S| + |T| \leq k \}$ and $A_k^+ = \{S|_1 : |S| \leq k \}$. For each $I \subseteq [n], |I| \leq k$, define $y^{(I)} \in A_k^+$ such that

$$y^{(I)}[S|_1] = \frac{\prod_{i \in S \setminus I} x_i}{\prod_{i \in I} x_i}$$

Note that in the case of $y^{(I)}[I|_1]$, the empty product is defined to evaluate to 1.

Next, we define $Y \in \mathbb{R}^{A_k^+ \times A_k^+}$ as

$$Y := \sum_{S \subseteq [n], |S| \leq k} \left( \prod_{i \in S} x_i (1 - x_i) \right) y^{(S)}(y^{(S)})^\top.$$
Note that $Y \succeq 0$. Now given $S, T \subseteq [n], |S|, |T| \leq k$, observe that
\[
Y[S|_1, T|_1] = \sum_{U \subseteq S \cap T} \left( \prod_{i \in U} x_i(1 - x_i) \right) \left( \prod_{i \in S \setminus U} x_i \right) \left( \prod_{i \in T \setminus U} x_i \right)
\]
\[
= \left( \prod_{i \in S \cup T} x_i \right) \left( \sum_{U \subseteq S \cap T} \left( \prod_{i \in U} (1 - x_i) \right) \left( \prod_{i \in (S \cup T) \setminus U} x_i \right) \right)
\]
\[
= \prod_{i \in S \cup T} x_i.
\]
Next, define $U \in \mathbb{R}^{A_k \times A_k^*}$ such that
\[
U^\top (e_{S|_1 \cap T|_0}) = \sum_{U : S \subseteq U \subseteq (S \cup T)} (-1)^{|U \setminus S|} e_{U|_1},
\]
for all disjoint $S, T \subseteq [n]$ such that $|S| + |T| \leq k$. We claim that $Y' := UYU^\top \in \tilde{S}A_k^+ (P)$.

First, notice that $Y'[\mathcal{F}, \mathcal{F}] = Y[\mathcal{F}, \mathcal{F}] = 1$, so (SA 1) holds. Next, given $\alpha = S|_1 \cap T|_0 \in A_k$, let $\hat{x} (Y'|_{\mathcal{F}, \alpha}) = \left( \begin{array}{c} Y'|_{[\mathcal{F}, \alpha]} \\ Y'|_{[\mathcal{F}, \alpha]} x_T^S \end{array} \right) \in K(P)$, by the assumption that $x_T^S \in P$. Thus, (SA 2) is satisfied as well. (SA 3) and (SA 3) are both ensured by the construction of $U$. Also, since $Y \succeq 0$, $Y' \succeq 0$ as well.

Finally, since $\hat{x}(Y'|_{\mathcal{F}}) = \hat{x}$, it follows that $x \in \tilde{S}A_k^+(P)$. \hfill \square

Corollary 16. Suppose $P \subseteq [0, 1]^n$ is a convex set such that, for all $x \in P$ and for all $I, J, I', J' \subseteq [n]$ such that $I \cup J = I' \cup J'$ and $|I| + |J| = k$,
\[
x_I^J \in P \iff x_{I'}^{J'} \in P.
\]
Then
\[
SA_+^k (P) = LS_0 (P) = \bigcap_{I \subseteq [n], |I| = k} \{ x : x_I^J \in P \}.
\]

The two results above generalize Theorem 4.1 and Corollary 4.2 in [GT01], respectively. Since $SA_+$ refines both $LS_+$ and $SA$, Corollary 16 immediately implies the following:

Corollary 17. Given $p \in \mathbb{R}$, let
\[
P(p) := \left\{ x \in [0, 1]^n : \sum_{i \in S} x_i + \sum_{i \notin S} (1 - x_i) \leq n - \frac{p + 1}{2}, \forall S \subseteq [n] \right\}.
\]
Then $SA_+^k (P(0)) = P(k)$, for all $k \in \{0, 1, \ldots, n\}$. In particular, the $SA_+$-rank of $P(0)$ is $n$.

One can apply the same argument used in Proposition 2 to show that $SA_+^k (P) \subseteq SA_+^k (P)$ in general. Thus, the $SA'_+$-rank of $P(0)$ is at least $\lceil \frac{n}{2} \rceil$. On the other hand, the proof of Proposition 23 can be adapted to show that the $SA'_+$-rank of any polytope contained in $[0, 1]^n$ is at most $\lceil \frac{n}{2} \rceil$. Thus, we see that in this case, $SA'_+$ requires roughly $\frac{n}{2}$ fewer rounds than $SA_+$ to show that $P(0)$ has an empty integer hull.
It was shown in [BZ04] that \( \text{BZ}^2(P(0)) = \emptyset = P(0) \) (implying \( \text{BZ}^2(P(0)) = \text{BZ}^2_+(P(0)) = \emptyset \)). However, since the run-time of BZ depends on the size of the system of inequalities describing \( P \) (which in this case is exponential in \( n \)), the relaxation generated by \( \text{BZ}^2 \) is not tractable. In contrast, note that it is easy to find an efficient separation oracle for \( P(0) \) (e.g. by observing \( x \in P(0) \iff \sum_{i=1}^n |x_i - \frac{1}{2}| \leq \frac{n-1}{2} \)), and thus one could optimize a linear function over, say, \( \text{SA}^k(P(0)) \) in polynomial time for any \( k = \Omega(1) \). The reader may refer to Figure 1 for a complete classification of operators that depend on the algebraic description of the input set \( P \), as opposed to those that only require a weak separation oracle.

5. Integrality gaps of lift-and-project relaxations

So far, we have been using the rank of a relaxation with respect to a lift-and-project operator as the measure of how far that relaxation is away from its integer hull. Another measure of the “tightness” of a relaxation that is commonly used and well studied is the integrality gap. Again, as the measure of how far that relaxation is away from its integer hull. Another measure of the optimizations in either Proposition 3 or 4. Then \( P \subseteq P' \subseteq \text{P} \) (which in this case is exponential in \( n \)). However, since the run-time of BZ depends on the size of the system of inequalities describing \( P \) (which in this case is exponential in \( n \)), the relaxation generated by \( \text{BZ}^2 \) is not tractable. In contrast, note that it is easy to find an efficient separation oracle for \( P(0) \) (e.g. by observing \( x \in P(0) \iff \sum_{i=1}^n |x_i - \frac{1}{2}| \leq \frac{n-1}{2} \)), and thus one could optimize a linear function over, say, \( \text{SA}^k(P(0)) \) in polynomial time for any \( k = \Omega(1) \). The reader may refer to Figure 1 for a complete classification of operators that depend on the algebraic description of the input set \( P \), as opposed to those that only require a weak separation oracle.

**Corollary 18.** Suppose \( P \subseteq [0,1]^n \), and two lift-and-project operators \( \Gamma_1, \Gamma_2 \) satisfy the conditions in either Proposition 3 or 4. Then

\[
\gamma_c(\Gamma_1(P)) \leq \gamma_c(\Gamma_2(P)),
\]

for all \( c \in \mathbb{R}^n \).

Next, we present another approach of obtaining an integrality gap result. Since in many optimization problems we are interested in computing the largest or smallest cardinality of a set among a given collection (e.g. the stable set problem and the max-cut problem), we are often optimizing in the direction of \( \bar{e} \). Moreover, we have seen that many hardness results have been achieved by highly symmetric combinatorial objects (e.g. the complete graph), which correspond to polytopes that have a lot of symmetries. Thus, the following ideas have been useful in the past and could continue to be useful.

We say that a compact convex set \( P \subseteq [0,1]^n \) is symmetric if there exists an \( n \)-by-\( n \) permutation matrix \( Q \) whose corresponding permutation on \( [n] \) has no cycles of length smaller than \( n \), such that \( \{Qx : x \in P\} \subseteq P \). Note that the reverse containment is implied by the definition, as \( Q^n = I \). Moreover, observe that if \( P \) is symmetric, then so is \( P_I \).

Next, we say that a lift-and-project operator \( \Gamma \) is symmetry preserving if given any symmetric, compact convex set \( P \), \( \Gamma(P) \) is also symmetric, compact and convex. All named operators mentioned in this paper are symmetry preserving. (In the case when \( \Gamma \) is one of the Bienstock–Zuckerberg variants, a symmetric algebraic description of \( P \) is required.) Then we have the following:

**Theorem 19.** Let \( P \subseteq [0,1]^n \) be a symmetric, compact and convex set, and let \( \Gamma \) be a symmetry preserving operator. Then, the integrality gaps of \( \gamma_c(\Gamma(P)) \) are attained by a nonnegative multiple of \( \bar{e} \).

**Proof.** First, we show that for any \( y \in \Gamma(P), (\frac{\bar{e}}{n}) \bar{e} \in P \). Let \( Q \) be a permutation matrix that certifies the symmetry of \( P \). Then given \( y \in \Gamma(P) \), we know that \( y, Qy, \ldots, Q^{n-1}y \in \Gamma(P) \), as \( \Gamma \).
preserves symmetry. Since $Q$ essentially permutes the $n$ coordinates of $P$ around in an $n$-cycle, we know that $\sum_{i=0}^{n-1} Q^i = J$, the all-ones matrix. By the convexity of $\Gamma(P)$,

$$\sum_{i=0}^{n-1} \frac{1}{n} (Q^i y) = \frac{1}{n} J y = \left( \frac{y^\top \bar{e}}{n} \right) \bar{e} \in \Gamma(P).$$

Now if $y$ is a point that attains the maximum integrality gap in the direction of $\bar{e}$, then we could use the above construction to obtain a multiple of $\bar{e}$ that achieves the same objective value. Hence, our claim follows.

Note that Theorem 19 immediately implies the following:

**Corollary 20.** Suppose $P \subseteq [0,1]^n$ is a symmetric, compact and convex set, and $\Gamma$ is a symmetry preserving operator. If $\ell \bar{e} \notin \Gamma(P)$ and $\ell_0 \bar{e} \in \Gamma(P)$ for some $\ell_0 < \ell$, then

$$\gamma_{\bar{e}}(\Gamma(P)) < \frac{\ell n}{\max \{ \sum_{i=1}^{n} x_i : x \in P_I \}}.$$

Of course, the $\gamma_{-\bar{e}}$ analogs of Proposition 19 and Corollary 20 can be obtained by essentially the same observations. Thus, we see that in many cases, it suffices to check whether a certain multiple of $\bar{e}$ belongs to $\Gamma(P)$ to obtain a bound on $\gamma_{\bar{e}}(\Gamma(P))$. This structure, when present, makes the analysis a lot easier, as often times we can apply the above symmetry-convexity argument to the certificate matrices in $\tilde{\Gamma}(P)$ as well, and identify many of the variables in the lifted space.

**References**


APPENDIX A. THE ORIGINAL BZ OPERATOR

In this section, we state the original BZ operator in our unifying language, and show that it is refined by BZ’.

The refinement step of $BZ^k$ coincides with $BZ^k$ — both operators derive $k$-small obstructions from the linear inequalities describing $P$, and use them to construct $O_k(P)$. Then $BZ^k$ defines its set of walls to be

$$W_k := \left\{ \bigcup_{i,j \in \ell : i \neq j} (O_i \cap O_j) : O_1, \ldots, O_\ell \in O_k, \ell \leq k + 1 \right\}.$$

Note that unlike for $BZ^k$, $BZ^k$ does not guarantee that the singleton sets are walls, and we will see that this could make a difference in performance. As for the tiers, $BZ^k$ defines them to be the sets of indices that can be written as the union of up to $k$ walls in $W_k$. Thus, $BZ^k$ only generates a polynomial size subset of the tiers used in $BZ^k$. Then the lifting step of $BZ^k$ (and $BZ^k_k$) can be described as follows:

1. Define $A'$ to be the set consisting of the following:
   - $\mathcal{F}$ and $i_{1,0}, \forall i \in [n]$.
   - Suppose $S := \bigcup_{i=1}^\ell W_i$ is a tier. Then we do the following:
     - For each $\ell$-tuple of sets, $(T_1, \ldots, T_\ell)$ such that $T_i \subseteq W_i$, $\forall i \in [\ell]$ and $\sum_{i=1}^\ell |T_i| \leq k$, include the set
   $$\left( \bigcup_{i=1}^\ell W_i \setminus T_i \right)|_1 \cap \left( \bigcup_{i=1}^\ell T_i \right)|_0.$$
   If $\sum_{i=1}^\ell |T_i| = k$ and $T_\ell \subset W_\ell$, then include the set
   $$\left( \bigcup_{i=1}^{\ell-1} W_i \setminus T_i \right)|_1 \cap \left( \bigcup_{i=1}^{\ell-1} T_i \right)|_0 \cap W_{\ell}|-|T_\ell|.$$

2. Let $BZ^k(P)$ denote the set of matrices $Y \in S^{A'}$ that satisfy all of the following conditions:
   (BZ1) $Y[\mathcal{F}, \mathcal{F}] = 1$.
   (BZ2) For any column $x$ of the matrix $Y$,
   (i) $0 \leq x_\alpha \leq x_\mathcal{F}$, for all $\alpha \in A'$.
   (ii) $\hat{x}(x) \in K(O_k(P))$.
   (iii) $x_{i_{1,0}} + x_{i_{1,0}} = x_\mathcal{F}$, for every $i \in [n]$.
   (iv) For each $\alpha \in A'$ in the form of $S|_1 \cap T|_0$, impose the inequalities
   $$x_{i_{1,0}} \geq x_\alpha, \forall i \in S;$$
   $$x_{i_{0,0}} \geq x_\alpha, \forall i \in T;$$
   $$\sum_{i \in S} x_{i_{1,0}} + \sum_{i \in T} x_{i_{1,0}} - x_\alpha \leq (|S| + |T| - 1)x_\mathcal{F}.$$
   (v) For each $\alpha \in A'$ in the form $S|_1 \cap T|_0 \cap U|_{<r}$, impose the inequalities
   $$x_{i_{1,0}} \geq x_\alpha, \forall i \in S;$$
   $$x_{i_{1,0}} \geq x_\alpha, \forall i \in T;$$
   $$\sum_{i \in U} x_{i_{1,0}} \geq (|U| - (r - 1))x_\alpha.$$
(vi) For each variable in the form \([\mathbf{14}]\), if \(|W_\ell| + \sum_{i=1}^{\ell-1} |T_i| \leq k\), impose
\[
\sum_{U \subseteq W_\ell} x(U_{\ell-1}w_i \setminus T_i)_1 \cap (U_{\ell-1}w_i \setminus T_i)_0 \cap (W_\ell \setminus U)_0
\]
\[\text{(22)}\]
Otherwise, define \(r := k - (\sum_{i=1}^{\ell-1} |T_i|)\), and impose
\[
\sum_{U \subseteq W_\ell, |U| \leq r} x(U_{\ell-1}w_i \setminus T_i)_1 \cap (U_{\ell-1}w_i \setminus T_i)_0 \cap (W_\ell \setminus U)_0
\]
\[\text{+} \quad x(U_{\ell-1}w_i \setminus T_i)_1 \cap (U_{\ell-1}w_i \setminus T_i)_0 \cap (W_\ell \setminus U)_0 \leq |W_\ell| - r
\]
\[\text{(23)}\]
(BZ 3) For all \(\alpha, \beta \in \mathcal{A}'\) such that \(\alpha \cap \beta = \emptyset\), or \(\alpha \cap \beta\) is contained in \(\mathcal{O}\) for some small obstruction \(\mathcal{O} \in \mathcal{O}_k\), \(\mathcal{Y}[\alpha, \beta] = 0\).
(BZ 4) For all \(\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathcal{A}'\) such that \(\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2\), \(\mathcal{Y}[\alpha_1, \beta_1] = \mathcal{Y}[\alpha_2, \beta_2]\).

(3) Define
\[
\mathcal{BZ}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \mathcal{BZ}^k(P), \hat{x}(Ye_F) = \hat{x} \right\},
\]
and
\[
\mathcal{BZ}^k_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \mathcal{BZ}^k_+(P), \hat{x}(Ye_F) = \hat{x} \right\},
\]
where \(\mathcal{BZ}^k_+(P) := \mathcal{BZ}^k(P) \cap \mathcal{S}_+\).

In \([\text{BZ04}]\), \(\mathcal{BZ}\) was defined so that the first relaxation in the hierarchy is \(\mathcal{BZ}^2(P)\), with \(\mathcal{BZ}^{n+1}(P)\) being the \(n\)th relaxation that is guaranteed to be \(P_1\). We have modified their definitions and presented their operators such that the relaxations are instead \(\mathcal{BZ}^1(P), \ldots, \mathcal{BZ}^n(P)\), to align them with the other named operators mentioned in this manuscript.

**Appendix B. Relationships among Variants of the BZ Operator, and some omitted Proofs**

Next, we show that \(\mathcal{BZ}'\) and \(\mathcal{BZ}'_+\) indeed refine their original counterparts.

**Proposition 21.** For every polytope \(P \subseteq [0,1]^n\) and integer \(k \geq 1\), \(\mathcal{BZ}^k(P) \subseteq \mathcal{BZ}^k_+(P)\) and \(\mathcal{BZ}^k_+(P) \subseteq \mathcal{BZ}^k_+(P)\).

**Proof.** It is apparent that every variable generated by \(\mathcal{BZ}^k\) is also generated by \(\mathcal{BZ}'^k\). The only nontrivial case is when \(\mathcal{BZ}^k\) generates a variable in the form
\[
\left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i\right)_1 \cap \left(\bigcup_{i=1}^{\ell-1} T_i\right)_0 \cap (W_\ell \setminus |W_\ell| - |T_\ell|)
\]
\[\text{(24)}\]
such that \(W_\ell\) is not disjoint from \(\bigcup_{i=1}^{\ell-1} W_i\). In this case if we define \(W' := W_\ell \setminus \bigcup_{i=1}^{\ell-1} W_i\), then the above is equivalent to \(\emptyset\) if \(|W'| \leq |T_\ell|\), and
\[
\left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i\right)_1 \cap \left(\bigcup_{i=1}^{\ell-1} T_i\right)_0 \cap (W' \setminus |W'| - |T_\ell|)
\]
on otherwise, which we know is generated by \(\mathcal{BZ}'^k\).

Also, the condition \((\text{BZ}'^3)\) is more easily triggered than \((\text{BZ} 3)\), and thus \(\mathcal{BZ}'\) forces more variables to be zero and is more restrictive. It is also not hard to see that the constraints \((\text{BZ}'^4)\)
implies their corresponding counterparts \([16] - [23]\) in BZ. Hence, we have \(BZ'^k(P) \subseteq BZ^k(P)\), and it follows readily that \(BZ'^k(P) \subseteq BZ^k(P)\) and \(BZ'^{+k}(P) \subseteq BZ^{+k}(P)\). \(\square\)

As Bienstock and Zuckerberg proved in [BZ04], the original BZ operator can efficiently solve many set covering type problems which require exponential effort to solve by previously known operators such as SA. However, since \(BZ^k\) does not ensure that it generates walls of small sizes, its tiers (which are unions of walls) could all be large, and the lifted set of variables \(A'\) does not necessarily contain \(A_k\) as in \(BZ'^k\). In fact, in some cases, \(BZ^k\) performs no better than one round of LS.

**Proposition 22.** Let \(p, q\) be positive integers such that \(1 \leq q < p\), and let
\[
P := \left\{ x \in [0, 1]^p : \sum_{i=1}^{p} x_i \leq q + \frac{1}{2} \right\}.
\]
If \((k + 1)(k + 2) \leq p - q\) and \(k + 1 \leq q\), then \(BZ^k(P) = LS(P)\) and \(BZ'^{+k}(P) = LS(P)\).

**Proof.** Since \(q + \frac{1}{2} > k + 1\), there are no \(k\)-small obstructions of size \(k + 1\) or less. Thus, \(S \subseteq [n]\) is a \(k\)-small obstruction if and only if \(|S| \geq p - (k + 1)\), which implies that every wall (and hence, every tier) has size at least \(p - (k + 1)^2\). If \(p - (k + 1)^2 - (k + 1) \geq q\), then we see that every tier is \(P\)-useless. The only remaining non-useless variables are \(F, i|_1\) and \(i|_0\) for all \(i \in [n]\). Thus, \(BZ^k(P) = LS(O_k(P))\) and \(BZ'^{+k}(P) = LS(O_k(P))\).

Furthermore, \(O_k(P) = P\) whenever \(k + 1 \leq p - q\), which is implied by \((k + 1)(k + 2) \leq p - q\). Thus, our claim follows. \(\square\)

Since \(LS(P) \subset P\) whenever \(P \neq P_1\), the above implies that one can construct examples in which \(LS(P) \subset BZ^k(P)\) for arbitrarily large \(k\). On the other hand, it is easy to obtain a lift-and-project operator that has the unique strength of BZ, while also refining the earlier operators (for instance, by simply taking \(\Gamma^k(P) = SA^k(P) \cap BZ^k(P)\)).

We can take this one step further. Recall that \(BZ'\) generates exponentially many variables in its lifted space, and thus does not admit a straightforward polynomial-time implementation. However, the number of variables generated becomes polynomial in \(n\) if we instead use the original BZ’s rule of generating tiers (i.e., defining \(S\) to be a tier if it is a union of up to \(k\) walls). Let \(BZ''\) denote this new operator. Then \(BZ''\) is just like the original BZ, except it has polynomially more variables, always ensures the singleton sets are walls, and imposes the condition (BZ’ 3) instead of the weaker (BZ 3). Also, just like (SA’ 4) and (SA’ 2), the condition (BZ’ 3) can be efficiently verified, given we have an efficient separation oracle for \(P\), and the condition is only checked polynomially many times. Replacing (BZ 3) with (BZ’ 3) boasts the advantage of eliminating the operator’s dependence on the set of obstructions in the lifting step, and allows us state the operator as a two-step process. Thus. if \(k = O(1)\) and we have a compact description of \(P\), then \(BZ''(P)\) is tractable. It is also not hard to see that \(BZ''\) refines both \(SA'\) and BZ. Moreover, the following is true:

**Proposition 23.** The \(BZ''\)-rank of \(P\) is at most \(\left\lceil \frac{n+1}{2} \right\rceil\), for all \(P \subseteq [0, 1]^n\).

**Proof.** Let \(Y \in BZ''^m(P)\) such that \(k \geq \frac{n+1}{2}\). We show that \(\hat{x}(Ye_F) \in K(P_t)\). Notice that \(BZ''(P)\) generates \(S := [k]\) is a tier (derived from \(k\) singleton-set walls), and we know by \([5]\) and the symmetry of \(Y\) that
\[
Ye_F = \sum_{T \subseteq S} Ye_{T|_1 \cap (S \setminus T)|_0}.
\]
In the remainder of this proof, we let $Y_T$ denote $Y_{e_{T|1 \cap (S \setminus T)|0}}$ to reduce cluttering. Note that since $|S| = k$, BZ$^{nk}$ does generate the variable $T|1 \cap (S \setminus T)|0$ for all $T \subseteq S$, and so $Y_T$ is well defined.

Next, we prove that $\hat{x}(Y_T) \in K(P_I)$ for every $T \subseteq S$. Then by (25), it follows that $\hat{x}(Y_{e_S}) \in K(P_I)$. For convenience, we let $\bar{S}$ denote $[n] \setminus S$. Notice that

$$ (Y_T)_{i|1} = \sum_{S' \subseteq S} (Y_T)_{S'|1 \cap (S \setminus S')|0} $$

by (5). Also, since $k \geq \frac{n+1}{2}$, $|\bar{S}| = n - k \leq k - 1$. Hence, $\{j\} \cup \bar{S}$ is a tier for all $j \in [n]$, and

$$ (Y_T)_{i|1} = \sum_{S' \subseteq S} (Y_T)_{(j \cup S')|1 \cap (\bar{S} \setminus S')|0}, \quad \forall j \in [n]. $$

Next, for all $T' \subseteq \bar{S}$, we define $Y_{T,T'} \in \mathbb{R}^{n+1}$ such that

$$ (Y_{T,T'})_i = \begin{cases} (Y_T)_{T'|1 \cap (S \setminus T')|0} & \text{if } i = 0 \text{ or } i \in T \cup T'; \\ 0 & \text{otherwise.} \end{cases} $$

From (26), (27), and the construction of $Y_{T,T'}$, we obtain that

$$ \hat{x}(Y_T) = \sum_{T' \subseteq \bar{S}} Y_{T,T'}, \quad \forall T \subseteq S. $$

Thus, it suffices to show that $Y_{T,T'} \in K(P_I)$, $\forall T \subseteq S, T' \subseteq \bar{S}$. This is obviously true if $Y_{T,T'}|0 = 0$. If $Y_{T,T'}|0 > 0$, then by (BZ' 3) we know that $(T \cup T')|1 \cap (\bar{S} \setminus T')|0 \cap P \neq \emptyset$. Since

$$ Y_{T,T'} = \begin{pmatrix} (Y_{T,T'})_0 \\ (Y_{T,T'})_{T \cup T'} \end{pmatrix}, $$

it follows that $Y_{T,T'} \in K(P_I)$, completing the proof. \qed

Likewise, we can define BZ$^n_+$ to be the positive semidefinite counterpart of BZ$^n$, and obtain a tractable operator that refines both SA$^\ell_+$ and BZ$^+$. Therefore, it follows that the SA$^\ell_+$-rank of any $P \subseteq [0,1]^n$ is also at most $\left\lceil \frac{n+1}{2} \right\rceil$. Moreover, observe that the essential ingredients used in the above proof are the presence of the variables in $A_{\lfloor n+1/2 \rfloor}$ in the lifted space and the condition (BZ' 3), which also applies for the SA$^{nk}_+$ relaxation for any $k \geq \frac{n+1}{2}$. Thus, the above proof can be slightly modified to show that the SA$^\ell_+$-rank of any polytope contained in $[0,1]^n$ is at most $\left\lceil \frac{n+1}{2} \right\rceil$. In contrast, we have seen in Corollary 17 an example in which the SA$^\ell_+$-rank is $n$.

While we do not have an example of a set whose BZ-rank exceeds $\left\lceil \frac{n+1}{2} \right\rceil$, we do have an instance in which BZ$^n$ outperforms BZ.

**Proposition 24.** Let $P := \left\{ x \in [0,1]^7 : \sum_{i=1}^7 2x_i \leq 7 \right\}$. Then

$$ y := (0.76, 0.76, 0.76, 0.3, 0.3, 0.3, 0.3)^\top \in \text{BZ}(P) \setminus \text{BZ}^n(P). $$

**Proof.** First, it is easy to see that $P_I = \left\{ x \in [0,1]^7 : \sum_{i=1}^7 x_i \leq 3 \right\}$, and $O_1(P) = P$. It can also be checked that $y \in \text{BZ}(P)$. We next show that BZ$^n$ cuts off $y$. First, the 1-small obstructions of $P$ is the collection of subsets of $[7]$ of size at least 5, and it is not hard to see that $O_1(P) = P$.

Since each wall is an intersection of up to two obstructions, every subset of $[7]$ of size between 3 and 5 is a wall. These sets are also exactly the tiers, as every tier is consisted of one wall in BZ$^n$. Suppose for a contradiction that there exists a certificate matrix $Y \in \text{BZ}^n(P)$ for $y$. Consider the tier $S := \{1, 2, 3\}$. By (10), we know that

$$ Y_{e_S} = Y_{e_S|1} + \sum_{i \in S} Y_{e_S(i)|1} + Y_{e_S|2}. $$
Since \( \hat{x}(Ye_{\alpha}) \in K(O_1(P)) = K(P) \) for all variables \( \alpha \in A' \), we know from (28) we can write
\[ \hat{x}(Ye_{F}) = z + w, \]
where \( z := \hat{x}(Ye_{S|_1}) \), and \( w \in K(P) \).

Now, applying (6) of \( S|_1 \) on the column \( Ye_{F} \), we obtain that
\[ Y_{[1|_1, F]} + Y_{[2|_1, F]} + Y_{[3|_1, F]} - Y_{[S|_1, F]} \leq (|S| - 1)Y_{[F, F]}. \]
Hence, \( z_0 = Y_{[F, S|_1]} = Y_{[S|_1, F]} \geq 3(0.76) - 2 = 0.28 \), and \( w_0 = 1 - z_0 \geq 0.72 \). We also know that \( \sum_{i=1}^{7} w_i \leq \frac{7}{2} w_0 \) (as \( w \in K(P) \)).

For \( j \in \{4, 5, 6, 7\} \), since \( j|_1 \cap S|_1 \cap P = \emptyset \), our strengthened rule (BZ'+3) requires that \( Y_{[j|_1, S|_1]} = 0 \) (this is what sets BZ'' apart from BZ in this example). Therefore, we have
\[ \sum_{i=1}^{7} z_i = \sum_{i=1}^{7} Y_{[i|_1, S|_1]} \leq 3Y_{[F, S|_1]} = 3z_0. \]
Thus, the inequality
\[ \sum_{i=1}^{7} x_i = \sum_{i=1}^{7} (z_i + w_i) \leq 3z_0 + \frac{7}{2} w_0 \leq 3(0.28) + \frac{7}{2} (0.72) = 3.36 \]
is valid for BZ''(P). However, \( \sum_{i=1}^{7} y_i = 3.48 \), which implies that \( y \notin BZ''(P) \).

Next, we remark that, in general, adding redundant inequalities to the system \( Ax \leq b \) could generate more obstructions and walls, and thus can improve the performance of BZ (and its variants). An example of this phenomenon is the following:

**Proposition 25.** Let \( G \) be the graph in Figure 4. Further let \( P \) be the set defined by the facets of \( FRAC(G) \) and \( P' \) be the system \( P \) with the additional (redundant) inequality
\[ \sum_{i \in V} x_i \leq 3. \]
Then
\[ BZ'_+(P) \supset BZ(P') = P_I. \]

**Proof.** For the first claim, notice that the obstructions generated by BZ'_+ are exactly the edge sets, so \( O_k(P) = (P) \). This also implies that all walls and tiers have size 1, so
\[ BZ'_+(P) = LS_+(O_k(P)) = LS_+(P) \neq P_I, \]
as it is shown in [LT03] that \( P \) has \( LS_+ \)-rank 2.

For the second claim, notice that with the additional inequality in \( P' \), all sets of size at least 4 are 1-small obstructions, and thus all sets of size 2 are walls (and hence tiers). In this case, \( BZ(P') \subseteq SA^2(P') = P_I. \)
Finally, we provide the proof to Theorem 8.

**Proof of Theorem 8.** Let $P := \text{FRAC}(K_n)$. We first prove the lower bound, by showing that all tiers generated by $BZ^k$ of size greater than $k + 1$ are $P$-useless. This, combined with Theorem 5, implies that $BZ^k(P) \supseteq \text{SA}^{2k+2}(O_k(P))$.

Since the set of $k$-small obstructions of $\text{FRAC}(K_n)$ is exactly $E$ for every $k \geq 1$, we see that $W_k = \{W \subseteq [n] : |W| \leq k + 1\}$ and $T_k = \{S \subseteq [n] : |S| \leq k(k + 1)\}$. Now if $S$ is any tier of size at least $k + 2$, we see that $(S \setminus T)|_1 \cap T|_0 \cap P = \emptyset$ for all $T \subseteq S$ such that $|T| \leq k$. This is because in such cases $|S \setminus T| \geq 2$, and there are no points in $P$ which contain at least two 1s. Thus, the only variables $\alpha$ associated with $S$ such that $\alpha \cap P \neq \emptyset$ take the form $(S \setminus (T \cup U))|_1 \cap T|_0 \cap U|_{[U]-(k-|T|)}$. However, in this case we know that $S \setminus (T \cup U)$ has size 0 or 1, and thus $\alpha \cap P$ is equal to either $F \cap P$ or $i|_1 \cap P$ for some $i \in [n]$. Therefore, all variables associated with $S$ are $P$-useless, and so the tier $S$ is $P$-useless.

Also, observe that $P = O_k(P)$ for any $k \geq 1$, and $P$ is known to have $\text{SA}$-rank $n - 2$. In fact, the matrix that certifies $\frac{1}{n-1} \bar{e} \in \text{SA}^{n-3}(P)$ also belongs to $\text{SA'}^{n-3}(P)$. Hence, the $\text{SA'}$-rank of $P$ is $n - 2$ as well. Thus, it follows that the $BZ'$-rank of $P$ is at least $\left\lceil \frac{n}{2} \right\rceil - 2$. Moreover, since $BZ'$ refines $BZ''$, it follows from Proposition 23 that $\text{FRAC}(G)$ has $BZ'$-rank at most $\left\lceil \frac{n+1}{2} \right\rceil$.

Finally, we turn to the $BZ$-rank of $\text{FRAC}(G)$. Again, $O_k = E$ for all $k \geq 1$. Therefore, in this case the conditions (BZ 3) and (BZ' 3) are equivalent. Since each vertex is incident with at least 2 edges, $BZ$ does generate all the singleton sets as walls. Thus, the $BZ$- and $BZ'$-rank of $\text{FRAC}(G)$ must coincide. \qed