Multi-row approaches to cutting plane generation

Laurent Poirrier

Montefiore Institute, ULg

Tuesday, December 18th, 2012
Example: The university is hiring

<table>
<thead>
<tr>
<th></th>
<th>Junior</th>
<th>Senior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teaching</td>
<td>40 hours</td>
<td>80 hours</td>
</tr>
<tr>
<td>Pay</td>
<td>$31</td>
<td>$45</td>
</tr>
<tr>
<td>Hire</td>
<td>at least one third</td>
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Have as many taught hours as possible, with a budget of $239.

\[
\begin{align*}
\text{max} & \quad 40 x_1 + 80 x_2 \\
\text{s.t.} & \quad 31 x_1 + 45 x_2 \leq 239 \\
& \quad x_1 \geq \frac{1}{2} x_2 \\
& \quad x_1, x_2 \geq 0 \\
& \quad x_1, x_2 \in \mathbb{Z}
\end{align*}
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\end{align*}
\]
$x_1 \geq 0$

$\geq$

$x_2 \geq 0$
$31x_1 + 45x_2 \leq 239$
\[ x_1 \geq \frac{1}{2} x_2 \]
Applications

- Scheduling (timetable building, machine tool switching, . . .)
- Bin-packing (chipset floor planning, . . .)
- Traveling Salesman Problem (ICs soldering and drilling)
- Discrete flow problems (power and energy distribution, . . .)
- Assignment
- Lot-sizing
- Transportation problems
- . . .

Most are NP-hard, and computationally difficult to solve.
Applications

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- Discrete flow problems (power and energy distribution, ...)
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- Transportation problems
- ...

Most are NP-hard, and computationally difficult to solve.
A Mixed Integer linear Programming problem

\[(\text{MIP}) \quad \begin{align*}
\min \quad & c^T x \\
\text{s.t.} \quad & Ax \geq b \\
\quad & x_j \in \mathbb{Z}, \text{ for } j \in J
\end{align*}\]
Solving MIPs: branch and bound

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\end{align*}\]

\(x_i^* \notin \mathbb{Z}\)
Solving MIPs: branch and bound

\[(MIP1)\]
\[\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad A x \geq b \\
& \quad x_i \leq \lfloor x^*_i \rfloor \\
& \quad x_j \in \mathbb{Z}, \text{ for } j \in J
\end{align*}\]

\[(MIP2)\]
\[\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad A x \geq b \\
& \quad x_i \geq \lceil x^*_i \rceil \\
& \quad x_j \in \mathbb{Z}, \text{ for } j \in J
\end{align*}\]
Solving MIPs: branch and bound

\begin{align*}
\text{(MIP1)} & \quad \min & c^T x \\
& \text{s.t.} & Ax \geq b \\
& & x_i \leq \lceil x_i^* \rceil \\
& & x_j \in \mathbb{Z}, \text{for } j \in J
\end{align*}

\begin{align*}
\text{(MIP2)} & \quad \min & c^T x \\
& \text{s.t.} & Ax \geq b \\
& & x_i \geq \lfloor x_i^* \rfloor \\
& & x_j \in \mathbb{Z}, \text{for } j \in J
\end{align*}
Cuts / Valid inequalities

(MIP) \( \min \ c^T x \) \\
\text{s.t.} \ A x \geq b \\
\text{for } x_j \in \mathbb{Z}, \text{ for } j \in J \\
\alpha x \geq 1
Cuts / Valid inequalities

(MIP) \[ \min \quad c^T x \]
\[ \text{s.t.} \quad A x \geq b \]
\[ x_j \in \mathbb{Z}, \text{for } j \in J \]

(cut) \[ \alpha x \geq 1 \]
## Why cut?

Most often, no cuts ↔ more cuts computing cuts each b&b node b&b nodes

<table>
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<tr>
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In practice, disabling cuts → $54 \times$ slower

(geometric mean over 719 instances [Bixby, Rothberg, 2007])
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In practice,

disabling cuts $\rightarrow 54\times$ slower

(geometric mean over 719 instances [Bixby, Rothberg, 2007]).
Example of cut

Let $x \in \mathbb{Z}_+^3$,

$$3x_1 + 4x_2 - 5x_3 \leq 4.5$$

\[ \Downarrow \]

$$3x_1 + 4x_2 - 5x_3 \leq 4$$
Example of cut

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\[
3x_1 + 4x_2 - 5x_3 \leq 4.5
\]

\[
\downarrow
\]

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3x_1 + 4x_2 - 5x_3 \leq 4
\]
Example of cut

Let $x \in \mathbb{Z}_+^3$,

$$3.4x_1 + 4.2x_2 - 4.6x_3 \leq 4.5$$

⇓

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Chvátal-Gomory cut
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Chvátal-Gomory cut
### What cuts?

<table>
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<tr>
<th>Disabled cut</th>
<th>Performance degradation</th>
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<tbody>
<tr>
<td>Gomory mixed-integer</td>
<td>2.52 ×</td>
</tr>
<tr>
<td>Mixed-integer rounding</td>
<td>1.83 ×</td>
</tr>
<tr>
<td>Knapsack cover</td>
<td>1.40 ×</td>
</tr>
<tr>
<td>Flow cover</td>
<td>1.22 ×</td>
</tr>
<tr>
<td>Implied bound</td>
<td>1.19 ×</td>
</tr>
<tr>
<td>Flow path</td>
<td>1.04 ×</td>
</tr>
<tr>
<td>Clique</td>
<td>1.02 ×</td>
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<tr>
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(geometric mean over 106 medium-sized instances [Bixby, Rothberg, 2007]).
A. Two-row cuts
A.1. Background
Single-row cuts

From one (re)formulation of the problem

\[
\begin{align*}
\text{(MIP)} \quad \min & \quad \bar{c}^T x \\
\text{s.t.} & \quad \bar{A} x \geq \bar{b} \\
& \quad x_j \in \mathbb{Z}
\end{align*}
\]

we extract one constraint \( \bar{A}_i x \geq \bar{b}_i \).

- Knowing that \( x_j \in \mathbb{Z} \), we construct a stronger inequality.
- In some cases, the cut can separate a specific point \( x^* \).
Single-row cuts

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\begin{align*}
\text{(MIP)} & \quad \min c^T x \\
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- In some cases, the cut can separate a specific point \( x^* \).
Two-row cuts

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\begin{align*}
\min & \quad \mathbf{c}^T \mathbf{x} \\
\text{s.t.} & \quad \mathbf{A} \mathbf{x} = \mathbf{b} \\
& \quad \mathbf{x} \geq 0 \\
& \quad \mathbf{x}_j \in \mathbb{Z}
\end{align*}
\]  

(MIP)

we extract two constraints

\[
\begin{align*}
x_1 + \sum_j a_{1j} s_j &= f_1 + x_2 + \sum_j a_{2j} s_j = f_2, \quad x_1, x_2 \in \mathbb{Z} \\
& \quad s_j \in \mathbb{R}_+
\end{align*}
\]

As a vector equation,

\[
(PI) \quad x = f + \sum_j r^j s_j, \quad x \in \mathbb{Z}^2, \quad s \in \mathbb{R}_+^n
\]

In case (MIP) describes a simplex tableau, \((x_{LP}^*, s_{LP}^*) = (f, 0)\).
Two-row cuts

From one (re)formulation of the problem

\[
\begin{align*}
\text{min} \quad & c^T x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0 \\
& x_j \in \mathbb{Z}
\end{align*}
\]

(MIP)

we extract \textbf{two} constraints

\[
\begin{align*}
x_1 + \sum_j \bar{a}_{1j} s_j &= f_1 \\
+ x_2 + \sum_j \bar{a}_{2j} s_j &= f_2,
\end{align*}
\]

\[x_1, x_2 \in \mathbb{Z}, \quad s_j \in \mathbb{R}_+\]

As a vector equation,

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\begin{align*}
(P_I) \quad x &= f + \sum_j r^j s_j, \\
x &\in \mathbb{Z}^2 \\
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In case (MIP) describes a simplex tableau, \((x^*_{LP}, s^*_{LP}) = (f, 0)\).
Two-row cuts

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x_1 + \sum_j \bar{a}_{1j} s_j &= f_1 \quad x_1, x_2 \in \mathbb{Z} \\
x_2 + \sum_j \bar{a}_{2j} s_j &= f_2 \quad s_j \in \mathbb{R}_+
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\]

(MIP)

we extract two constraints

\[
\begin{align*}
x_1 + x_2 + \sum_j \bar{a}_{1j} s_j &= f_1, & x_1, x_2 \in \mathbb{Z} \\
+ x_2 + \sum_j \bar{a}_{2j} s_j &= f_2, & s_j \in \mathbb{R}_+
\end{align*}
\]

As a vector equation,

\[(P_I) \quad x = f + \sum_j r^j s_j, \quad x \in \mathbb{Z}^2 \quad s \in \mathbb{R}_+^n\]

In case (MIP) describes a simplex tableau, \((x_{LP}^{*}, s_{LP}^{*}) = (f, 0)\).
A.2. Problem statement
The two-row model

\[ x = f + \sum_j r^j s_j \]
\[ x \in \mathbb{Z}^2 \]
\[ s_j \geq 0 \]
The two-row model

\[ x = f + \sum_j r^j s_j \]

\[ x \in \mathbb{Z}_2 \]

\[ s_j \geq 0 \]

Example:

\[ s = \left( \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

\[ x = f + \frac{1}{2} r^1 + \frac{1}{2} r^5 + \frac{1}{2} r^4 + \frac{1}{2} r^3 \]

\[ x = f + \frac{1}{2} r^4 + \frac{1}{12} r^2 \]
The two-row model

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The two-row model

\[ x = f + \sum_j r^j s_j \]
\[ x \in \mathbb{Z}^2 \]
\[ s_j \geq 0 \]

An inequality of the form

\[ \alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1 \]

with \( \alpha_i \geq 0 \), cuts off

\[ \text{interior}(L_\alpha) \]

in the \( x \) space

where \( v^i = f + \frac{1}{\alpha_i} r^i \).
The two-row model

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where \( v^i = f + \frac{1}{\alpha_i} r^i \).
Validity: The linear programming intuition

Given $\bar{x} \in \mathbb{Z}^2$, we want that

$$\forall s \in \mathbb{R}^n_+ : \bar{x} = f + Rs, \quad \alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1$$

i.e. we want

$$\min \alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1$$
$$\text{s.t. } Rs = \bar{x} - f$$
$$s \geq 0$$

therefore we need

$$\forall i, j, s_i^{\bar{x}}, s_j^{\bar{x}} : \bar{x} = f + s_i^{\bar{x}} r^i + s_j^{\bar{x}} r^j,$$
$$s_i^{\bar{x}} \alpha_i + s_j^{\bar{x}} \alpha_j \geq 1.$$
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s.t. $Rs = \bar{x} - f$

$s \geq 0$

therefore we need

$$\forall i, j, s^x_i, s^x_j : \bar{x} = f + s^x_i r^i + s^x_j r^j, \quad s^x_i \alpha_i + s^x_j \alpha_j \geq 1.$$
Validity: The linear programming intuition

Given \( \overline{x} \in \mathbb{Z}^2 \), we want that

\[
\forall s \in \mathbb{R}^n_+ : \overline{x} = f + Rs, \quad \alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1
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i.e. we want

\[
\min \alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1
\]

s.t. \( Rs = \overline{x} - f \)

\( s \geq 0 \)

therefore we need

\[
\forall i, j, s_i^\overline{x}, s_j^\overline{x} : \overline{x} = f + s_i^\overline{x} r^i + s_j^\overline{x} r^j, \quad s_i^\overline{x} \alpha_i + s_j^\overline{x} \alpha_j \geq 1.
\]
Given $\bar{x} \in \mathbb{Z}^2$,

for all $i, j : \bar{x} \in f + \text{cone}(r^i, r^j),$

$$s_{i}^{\bar{x}} \alpha_i + s_{j}^{\bar{x}} \alpha_j \geq 1,$$

with $s_{i}^{\bar{x}}, s_{j}^{\bar{x}} : \bar{x} = f + s_{i}^{\bar{x}} r^i + s_{j}^{\bar{x}} r^j.$
Lattice-free sets – the geometrical intuition

Given $\overline{x} \in \mathbb{Z}^2$,

for all $i, j$:

$$\overline{x} \in f + \text{cone}(r^i, r^j),$$

$$s^\overline{x}_i \alpha_i + s^\overline{x}_j \alpha_j \geq 1,$$

with $s^\overline{x}_i, s^\overline{x}_j : \overline{x} = f + s^\overline{x}_i r^i + s^\overline{x}_j r^j$. 
Lattice-free sets – the geometrical intuition

Given $\overline{x} \in \mathbb{Z}^2$, for all $i, j : \overline{x} \in f + \text{cone}(r^i, r^j)$,

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with $s^x_i, s^x_j : \overline{x} = f + s^x_i r^i + s^x_j r^j$. 


Given $\bar{x} \in \mathbb{Z}^2$, for all $i, j : \bar{x} \in f + \text{cone}(r^i, r^j)$,  

$$s_i^{\bar{x}} \alpha_i + s_j^{\bar{x}} \alpha_j \geq 1,$$

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Lattice-free sets – the geometrical intuition
Given \( \bar{x} \in \mathbb{Z}^2 \),

for all \( i, j : \bar{x} \in f + \text{cone}(r^i, r^j) \),

\[ s_i^{\bar{x}} \alpha_i + s_j^{\bar{x}} \alpha_j \geq 1, \]

with \( s_i^{\bar{x}}, s_j^{\bar{x}} : \bar{x} = f + s_i^{\bar{x}} r^i + s_j^{\bar{x}} r^j \).
Lattice-free sets – the intuition, for all $x$

For all $x \in \mathbb{Z}^2$,

for all $i, j : x \in f + \text{cone}(r^i, r^j)$,

$$s^x_i \alpha_i + s^x_j \alpha_j \geq 1,$$

with $s^x_i, s^x_j : x = f + s^x_i r^i + s^x_j r^j$. 
Lattice-free sets – the intuition, for every cone

For all $i, j$,

for all $x \in \mathbb{Z}^2 \cap (f + \text{cone}(r^i, r^j))$,

$$s_i^x \alpha_i + s_j^x \alpha_j \geq 1,$$

with $s_i^x, s_j^x : x = f + s_i^x r^i + s_j^x r^j$. 

$x_2$

$x_1$
Lattice-free sets – the set $\mathcal{X}_{ij}$

For all $i, j$,

for all $x \in \mathcal{X}_{ij}$,

$$s^x_i \alpha_i + s^x_j \alpha_j \geq 1,$$

with $s^x_i, s^x_j : x = f + s^x_i r^i + s^x_j r^j$.

▶ we can restrict $x \in \mathbb{Z}^2$ to

$x \in \mathcal{X}_{ij}$ where $\mathcal{X}_{ij}$ is the set of the vertices of

$\mathbb{Z}^2 \cap (f + \text{conv}(r^i, r^j))$. 
Let $P \subseteq \mathbb{R}^N$ be a radial polyhedron and $Q \subseteq \mathbb{R}^N$ its polar. There is a correspondance between

- Extreme point $\overline{x} \in P$ and Facet of $Q$: $\overline{x}^T a \geq 1$
- Extreme ray $\overline{x} \in P$ and Facet of $Q$: $\overline{x}^T a \geq 0$
- Facet of $P$: $\overline{a}^T x \geq 1$ and Extreme point $\overline{a} \in Q$
- Facet of $P$: $\overline{a}^T x \geq 0$ and Extreme ray $\overline{a} \in Q$
Polarity, applied

- We have a polyhedron
  \[ \text{conv} (P_I) = \text{conv} \left( \left\{ (x, s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ \mid x = f + \sum_j r^j s_j \right\} \right) . \]
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| Extreme point \( \bar{x} \in \text{conv} (P_I) \) | Facet of \( Q \): \( \bar{x}^T \alpha \geq 1 \) |
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Finding facets of $\text{conv } P_I$

The polar of $\text{conv}(P_I)$ is

$$Q = \{ \alpha \in \mathbb{R}^n_+ | \forall i, j, \forall x \in X_{ij}, \ s^x_i \alpha_i + s^x_j \alpha_j \geq 1 \}.$$ 

We find facets of $\text{conv}(P_I)$ by choosing an objective function $c^T \alpha$ and optimizing over $Q$:

$$\begin{align*}
\text{min} & \quad c^T \alpha \\
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For each cone, compute integer hull.
For each vertex, write one constraint.

1. Cones: quadratic in the number of rays.
2. Vertices: polynomial (but possibly large) number in each cone.
Complexity of writing the polar (1)

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▶ What is \( Q \setminus \overline{Q} \)?
The complexity of the polar – the theory

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Corollary

If \( c > 0 \), \[ \min_{\text{s.t. } \alpha \in Q} c^T \alpha \] and \[ \min_{\text{s.t. } \alpha \in \overline{Q}} c^T \alpha \] share the same set of optimal solutions.

If \( c_i < 0 \), then \[ \min_{\text{s.t. } \alpha \in Q} c^T \alpha \] is unbounded.
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A.4. Results
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- We have a fast separation for two-row cuts.
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But

- they do not close much more gap than two-row intersection cuts from split sets.
B. Separation over arbitrary mixed-integer sets
Motivations

- We want to test stronger relaxations
- We still want exact separation
B.1. Separation method
Problem

Given

- a relaxation $P$ of mixed-integer set in $\mathbb{R}^n$,
- a point $x^* \in \mathbb{R}^n$,

find $(\alpha, \alpha_0) \in \mathbb{R}^{n+1}$ such that

$$\alpha^T x \geq \alpha_0$$

is a valid inequality for $P$ that separates $x^*$,

or show that $x^* \in \text{conv}(P)$. 
Problem

Given

- a relaxation $P$ of mixed-integer set in $\mathbb{R}^n$,
- a point $x^* \in \mathbb{R}^n$,

find $(\alpha, \alpha_0) \in \mathbb{R}^{n+1}$ such that

$$\alpha^T x \geq \alpha_0$$

is a valid inequality for $P$ that separates $x^*$,

or show that $x^* \in \text{conv}(P)$. 

Solve the optimization problem

\[
\begin{align*}
\min & \quad x^* T \alpha \\
\text{s.t.} & \quad x^T \alpha \geq \alpha_0 \quad \text{for all} \ x \in P \\
& \quad \langle \text{norm.} \rangle
\end{align*}
\]

(Sep. LP)

Let \((\bar{\alpha}, \bar{\alpha}_0)\) be an optimal solution.

If \(x^T \bar{\alpha} < \bar{\alpha}_0\), then \((\bar{\alpha}, \bar{\alpha}_0)\) separates \(x^*\).

If \(x^T \bar{\alpha} \geq \bar{\alpha}_0\), then \(x^* \in \text{conv}(P)\).
General framework

Solve the optimization problem

\[
\begin{align*}
\min & \quad x^T \alpha \\
\text{s.t.} & \quad x^T \alpha \geq \alpha_0 \quad \text{for all } x \in P \\
& \quad \langle \text{norm.} \rangle
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(Sep. LP)

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If \(x^T \bar{\alpha} \geq \bar{\alpha}_0\), then \(x^* \in \text{conv}(P)\).
Row generation

1. Consider the relaxation of the separation problem

\[
\begin{align*}
\min & \quad x^T \alpha \\
\text{s.t.} & \quad x^T \alpha \geq \alpha_0 \quad \text{for all } x \in S \subseteq P \\
\end{align*}
\]  \text{ (master)}

Let \((\bar{\alpha}, \bar{\alpha}_0)\) be an optimal solution.

2. Now solve the MIP

\[
\begin{align*}
\min & \quad \bar{x}^T x \\
\text{s.t.} & \quad x \subseteq P \\
\end{align*}
\]  \text{ (slave)}

and let \(x^p\) be a finite optimal solution.

If \(\bar{x}^T x^p \geq \bar{\alpha}_0\), then \((\bar{\alpha}, \bar{\alpha}_0)\) is valid for \(P\).

If \(\bar{x}^T x^p < \bar{\alpha}_0\), then \(S := S \cup \{x^p\}\).
Row generation

1. Consider the relaxation of the separation problem

\[
\begin{align*}
\min & \quad x^* T \alpha \\
\text{s.t.} & \quad x^T \alpha \geq \alpha_0 \quad \text{for all } x \in S \subseteq P \quad \text{(master)} \\
\end{align*}
\]

Let \((\bar{\alpha}, \bar{\alpha}_0)\) be an optimal solution.

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\min & \quad \bar{\alpha}^T x \\
\text{s.t.} & \quad x \subseteq P \quad \text{(slave)} \\
\end{align*}
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Row generation

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\text{min} & \quad x^T \alpha \\
\text{s.t.} & \quad x^T \alpha \geq \alpha_0 \quad \text{for all } x \in S \subseteq P \\
\end{align*}
\]

(master)

\(<\text{norm.}>\)

Let \((\bar{\alpha}, \bar{\alpha}_0)\) be an optimal solution.

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If \(\bar{\alpha}^T x^p < \bar{\alpha}_0\), then \(S := S \cup \{x^p\}\).
Computational example

Instance: bell13a
Constraints: 123
Variables: 133 (71 integer: 32 general, 39 binaries)
Models: 82 five-row models read from an optimal tableau

Cuts: 37 (17 tight at the end)
Gap closed: 59.02% (from 39.03% by GMIs)

| Time:     | 1615.70s |
| Iterations: | 107647  |
Two-phases: Phase one

$x^*$ between bounds

$x : \begin{cases} x_B \\ x_N \end{cases}$

fix to bound

$x^*$ at bounds

$\alpha : \begin{cases} \alpha_B \\ \alpha_N \end{cases}$
Two-phases: Phase one

Two phases:

1. \( x^* \) between bounds
2. \( x^* \) at bounds

\[ x : \begin{cases} x_B & \text{fix to bound} \\ x_N \end{cases} \]

\[ \alpha : \begin{cases} \alpha_B \\ \alpha_N \end{cases} \]
Two-phases: Phase two

$x^*$ between bounds

$x : \begin{cases} x_B & x^* \text{ at bounds} \\ x_N & \end{cases}$

$x^*$ at bounds

$\alpha : \begin{cases} \alpha_B & \text{fixed} \\ \alpha_N & \text{lift} \end{cases}$

fix to bound
Computational example (2-phases)

(bell3a, 82 five-row models, 37 cuts, 59.02% gc)

<table>
<thead>
<tr>
<th></th>
<th>original</th>
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<tbody>
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<td>Time:</td>
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Lifting binary variables

\[ x^* \text{ between bounds} \quad \quad x^* \text{ at bounds} \]

\[
\begin{array}{c}
 x : \\
 x_B \quad x_{\text{Nbin}} \quad x_{\text{N'}}
\end{array}
\]

\[
\begin{array}{c}
 \alpha : \\
 \alpha_B \quad \alpha_{\text{Nbin}} \quad \alpha_{\text{N'}}
\end{array}
\]

\[ \text{fixed} \]
Lifting binary variables

\[ \begin{align*}
  x^* \text{ between bounds} & \quad x^* \text{ at bounds} \\
  \{ & \text{binary} \\
  x : & x_B \quad x_{N\text{bin}} \quad x_{N'} \\
  \alpha : & \alpha_B \quad \alpha_{N\text{bin}} \quad \alpha_{N'} \\
  \{ & \text{fixed} \quad \text{lift}
\end{align*} \]
Lifting binary variables

\[ x^* \text{ between bounds} \quad x^* \text{ at bounds} \]

\[ x : \quad x_B \quad x_{Nbin} \quad x_N' \]

\[ \alpha : \quad \alpha_B \quad \alpha_{Nbin} \quad \alpha_N' \]

binary

fixed

fixed
Computational example (lifting binaries)

(bell3a, 82 five-row models, 37 cuts, 59.02%gc)

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(bell3a, 82 five-row models, 37 cuts, 59.02\%gc)

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</tbody>
</table>
### Sequential phase-2 ("phase-S")

$x^*$ between bounds

$x^*$ at bounds

\[
\begin{array}{c}
\text{binary} \\
\hline
x : \\
\hline
x_B & x_{N\text{bin}} & x_k & x_{N''}
\end{array}
\]

\[
\begin{array}{c}
\text{fixed} \\
\hline
\alpha : \\
\hline
\alpha_B & \alpha_{N\text{bin}} & \alpha_k & \alpha_{N''}
\end{array}
\]

(fixed to bnd)

zero
Sequential phase-2 ("phase-S")

\[ x^* \text{ between bounds} \quad \text{and} \quad x^* \text{ at bounds} \]

\[
\begin{align*}
x & : \\
& \quad x_B \quad x_{Nbin} \quad x_k \quad x_{N''}
\end{align*}
\]

\[
\begin{align*}
\alpha & : \\
& \quad \alpha_B \quad \alpha_{Nbin} \quad \alpha_k \quad \alpha_{N''}
\end{align*}
\]

\( \text{binary} \)

(fixed to bnd)

\( \text{fixed} \)

\( \text{zero} \)
Sequential phase-2 ("phase-S")

\[ x^* \text{ between bounds} \quad \{ \text{binary} \} \quad x^* \text{ at bounds} \]

\[ x : \begin{array}{c} \alpha_B \\ \alpha_{N_{bin}} \\ \alpha_k \\ \alpha_{N''} \end{array} \]

\[ (\text{fixed to bnd}) \]

\[ \alpha : \begin{array}{c} \alpha_B \\ \alpha_{N_{bin}} \\ \alpha_k \\ \alpha_{N''} \end{array} \]

\[ \{ \text{fixed} \} \quad \{ \text{lift} \} \quad \{ \text{zero} \} \]
Sequential phase-2 ("phase-S")

\[ x^* \text{ between bounds} \quad \begin{array}{c}
\alpha^* \text{ at bounds} \\
\text{binary}
\end{array} \]

\[ x : \begin{array}{c}
\alpha \quad \alpha_N bin \quad \alpha_k \quad \alpha_N''
\end{array} \]

\[ x : \begin{array}{c}
\alpha \quad \alpha_N bin \quad \alpha_k \quad \alpha_N''
\end{array} \]

\[ \begin{array}{c}
\alpha \text{ fixed} \\
\alpha_N'' \text{ fixed}
\end{array} \]

\[ x \text{ fixed} \quad \alpha_N'' \text{ fixed} \quad \text{zero} \]
Computational example (phase S)

(bell3a, 82 five-row models, 37 cuts, 59.02% gc)

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Computational example (phase S)

(bell3a, 82 five-row models, 37 cuts, 59.02%gc)

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Computational example (solver tricks)

(bell3a, 82 five-row models, 37 cuts, 59.02%gc)

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<tr>
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<td>4.65s</td>
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Computational example (solver tricks)

(bell13a, 82 five-row models, 37 cuts, 59.02\%gc)

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</table>
Computational example (summary)

(bell3a, 82 five-row models, 37 cuts, 59.02%gc)

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<tr>
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<td>29×</td>
<td>1.26×</td>
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<td>2497</td>
<td>2497</td>
</tr>
<tr>
<td></td>
<td>43×</td>
<td>10×</td>
<td>9×</td>
<td>1</td>
<td>1</td>
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</table>
B.2. Application to two-row relaxations
# Two-row intersection cuts + strengthening

<table>
<thead>
<tr>
<th></th>
<th>basic</th>
<th>nonbasic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \in \mathbb{Z} ) bnd.</td>
<td>( \in \mathbb{Z} ) bnd.</td>
<td></td>
</tr>
<tr>
<td>( P_I )</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>( S )-free</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>lifting</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>( P_{IU} )</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>full 2-row</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

\( ✓ \): keep

\( B \): keep binding

\( × \): drop
## Two-row intersection cuts + strengthening

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<tr>
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<td>$\in \mathbb{Z}$</td>
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<tr>
<td>$P_I$</td>
<td>$\checkmark$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$S$-free</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>lifting</td>
<td>$\checkmark$</td>
<td>$\times$</td>
</tr>
<tr>
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</tr>
<tr>
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<td>$\checkmark$</td>
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$\checkmark$: keep  
B: keep binding  
$\times$: drop
# Two-row intersection cuts + strengthening

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<td>√</td>
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<td>×</td>
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<tr>
<td>$P_{IU}$</td>
<td>√</td>
<td>×</td>
</tr>
<tr>
<td>full 2-row</td>
<td>√</td>
<td>√</td>
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</table>

- **√**: keep
- **B**: keep binding
- **×**: drop
Two-row intersection cuts + strengthening

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$\checkmark$: keep
B: keep binding
$\times$: drop
## Two-row intersection cuts + strengthening

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<td>$P_I$</td>
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<td>× B</td>
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<td>× B</td>
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<td>√ B</td>
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- **√**: keep
- **B**: keep binding
- **×**: drop
Two-row intersection cuts + strengthening

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$\surd$: keep  
B: keep binding  
$\times$: drop
Two-row intersection cuts and strengthenings

51 common instances:

<table>
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<tr>
<th></th>
<th>cuts</th>
<th>gc%</th>
<th>exact</th>
</tr>
</thead>
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<tr>
<td>GMI</td>
<td>28.240</td>
<td>22.46%</td>
<td>all</td>
</tr>
<tr>
<td>$P_I$</td>
<td>29.420</td>
<td>27.65%</td>
<td>42</td>
</tr>
<tr>
<td>S-free</td>
<td>38.380</td>
<td>30.22%</td>
<td>29</td>
</tr>
<tr>
<td>lifting</td>
<td>22.700</td>
<td>27.35%</td>
<td>10</td>
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<tr>
<td>$P_{IU}$</td>
<td>42.640</td>
<td>28.56%</td>
<td>25</td>
</tr>
<tr>
<td>full 2-row</td>
<td>55.500</td>
<td>35.66%</td>
<td>22</td>
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Two-row intersection cuts and strengthenings

51 common instances:

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<td>29</td>
</tr>
<tr>
<td>lifting</td>
<td>22.700</td>
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<td>10</td>
</tr>
<tr>
<td>$P_{IU}$</td>
<td>42.640</td>
<td>28.56%</td>
<td>25</td>
</tr>
<tr>
<td>full 2-row</td>
<td>55.500</td>
<td>35.66%</td>
<td>22</td>
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Two-row intersection cuts and strengthenings

51 common instances:

<table>
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<tr>
<th></th>
<th>cuts</th>
<th>gc%</th>
<th>exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMI</td>
<td>28.240</td>
<td>22.46%</td>
<td>all</td>
</tr>
<tr>
<td>$P_I$</td>
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</tr>
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## Two-row intersection cuts and strengthenings

15 common instances:

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<tr>
<th></th>
<th>cuts</th>
<th>gc%</th>
<th>exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMI</td>
<td>20.667</td>
<td>26.541</td>
<td>all</td>
</tr>
<tr>
<td>$P_I$</td>
<td>20.933</td>
<td>33.535</td>
<td>all</td>
</tr>
<tr>
<td>S-free</td>
<td>25.400</td>
<td>35.229</td>
<td>all</td>
</tr>
<tr>
<td>$P_{IU}$</td>
<td>36.600</td>
<td>36.257</td>
<td>all</td>
</tr>
<tr>
<td>full 2-row</td>
<td>57.267</td>
<td>43.956</td>
<td>all</td>
</tr>
</tbody>
</table>
Two-row intersection cuts and strengthenings

7 common instances:
[bell5, blend2, egout, khb05250, misc03, misc07, set1ch]

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<th>exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMI</td>
<td>25.571</td>
<td>24.744</td>
<td>all</td>
</tr>
<tr>
<td>$P_I$</td>
<td>25.143</td>
<td>33.641</td>
<td>all</td>
</tr>
<tr>
<td>S-free</td>
<td>28.714</td>
<td>33.836</td>
<td>all</td>
</tr>
<tr>
<td>lifting</td>
<td>25.571</td>
<td>33.716</td>
<td>all</td>
</tr>
<tr>
<td>$P_{IU}$</td>
<td>47.857</td>
<td>37.531</td>
<td>all</td>
</tr>
<tr>
<td>full 2-row</td>
<td>48.000</td>
<td>37.583</td>
<td>all</td>
</tr>
</tbody>
</table>
Bases

- We depend on a specific optimal basis
- Will the gap closed by two-row cuts survive more GMIs?
Bases

- We depend on a specific optimal basis
- Will the gap closed by two-row cuts survive more GMIs?
Bases

- We depend on a specific optimal basis
- Will the gap closed by two-row cuts survive more GMIs?
Relax and cut: results

43 common instances:

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</thead>
<tbody>
<tr>
<td>GMI</td>
<td>24.814</td>
<td>23.282</td>
<td>all</td>
</tr>
<tr>
<td>2-row i.c.</td>
<td>31.884</td>
<td>28.838</td>
<td>42</td>
</tr>
<tr>
<td>full 2-row</td>
<td>62.140</td>
<td>36.080</td>
<td>22</td>
</tr>
<tr>
<td>relax&amp;cut GMI</td>
<td>60.372</td>
<td>34.970</td>
<td>all</td>
</tr>
<tr>
<td>relax&amp;cut 2-row i.c.</td>
<td>63.163</td>
<td>41.951</td>
<td>37</td>
</tr>
<tr>
<td>relax&amp;cut full 2-row</td>
<td>76.767</td>
<td>47.277</td>
<td>12</td>
</tr>
</tbody>
</table>
More rows: Computing time

instances with result, and instances with exact separation

geometric mean of time (on 42 common instances)
More rows: Gap closed

number of cuts generated (on 42 common instances)

average %gc (on 42 common instances)
Overall summary

- a (quick) two-row intersection cut separator
  - assessment: strength of the two-row model
- a (slow) generic arbitrary-MIP cut separator
  - assessment: strength of multi-row model and variants
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Conclusions

Multi-row cuts:

- Number of rows: few or almost all
- Intersection cuts: need to apply all strengthenings
Conclusions

Multi-row cuts:

- Number of rows: few or almost all

- Intersection cuts: need to apply all strengthenings
The integer hull

Adding all valid inequalities for (MIP), we obtain:

\[
\text{conv}\{x : x \in (\text{MIP})\}
\]

In theory: as hard as solving (MIP)

In practice: much harder
The integer hull

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In theory: as hard as solving \((MIP)\)

In practice: much harder
Can we avoid the integer hulls $X_{ij}$?

$$\overline{Q} = \{ \alpha \in \mathbb{R}^n_+ \mid \forall i, \forall x \in X_{i,i+1}, \forall i : r^i \in \text{cone}(r^{i-1}, r^{i+1}),$$

$$s_i^x \alpha_i + s_{i+1}^x \alpha_{i+1} \geq 1,$$

$$\alpha_i \leq \lambda_{i-1}^i \alpha_{i-1} + \lambda_{i+1}^i \alpha_{i+1} \}$$

$$\overline{Q}(S) = \{ \alpha \in \mathbb{R}^n_+ \mid \forall i, \forall x \in S \cap (f + \text{cone}(r^i, r^{i+1})), \forall i : r^i \in \text{cone}(r^{i-1}, r^{i+1}),$$

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with $S \subset \mathbb{Z}^2$. 
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with $S \subset \mathbb{Z}^2$. 
Separation algorithm

\[ S := S_0 \]

\[
\text{do \{ \\
\quad \alpha := \arg\min \ c^T \alpha \\
\quad \text{s.t. } \alpha \in \overline{Q}(S) \\
\}} \]

if \( \alpha \in \overline{Q} \)

OK, valid cut, exit.

else

Find a constraint of \( \overline{Q} \) violated by \( \alpha \).

Add constraints to \( S \).
Separation algorithm

\[ S := S_0 \]
\begin{align*}
&\text{do } \{ \\
&\qquad \alpha := \text{argmin } c^T \alpha \\
&\quad \text{s.t. } \alpha \in \overline{Q}(S) \\
&\quad \text{if } L_\alpha \text{ is lattice-free} \\
&\quad \quad \text{OK, valid cut, exit.} \\
&\quad \text{else} \\
&\quad \quad \text{Find } x \in \mathbb{Z}^2 \cap \text{interior}(L_\alpha). \\
&\quad \quad \text{Add } x \text{ to } S.
&\}\end{align*}
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\} \]
Integer pair extension
Integer pair extension
Integer pair extension
Integer pair extension
The oracle

Find an integer point in $\text{interior}(L_\alpha)$ or prove that $L_\alpha$ is lattice-free.

- possible in polynomial time for any fixed dimension $d$ (Barvinok’s algorithm)
- but $d = 2$
- we know $S \cap L_\alpha$
- closed-form formula?
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The oracle: $\text{conv}(T)$

Find an integer point in $\text{interior}(L_\alpha)$ or prove that $L_\alpha$ is lattice-free.

1. Consider the convex hull $\text{conv}(T)$ where $T := S \cap \text{boundary}(L_\alpha)$.
   - triangularize $\text{conv}(T)$
   - find integer points on integer segments and integer triangles
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The oracle: \( \text{conv}(T) \), continued

Let \( \Delta \) be \( \text{conv}(0, u, v) \) with \( u, v \in \mathbb{Z} \) and \( \gcd(u_1, u_2) = \gcd(v_1, v_2) = 1 \).

\[
\left\{ \begin{array}{l}
\frac{\lambda}{\det([u|v])} u + \frac{\mu}{\det([u|v])} v : \lambda, \mu \in \mathbb{Z}_+, 0 < \lambda + \mu < \det([u|v])
\end{array} \right\}
\]

Prop.: \( \Delta \) has an interior lattice point with \( \mu = 1 \), or is lattice-free.

It is enough to solve the diophantine system

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\left\{ \begin{array}{l}
\lambda u_1 + v_1 = k_1 \det([u|v]) \\
\lambda u_2 + v_2 = k_2 \det([u|v])
\end{array} \right\}, \lambda, k_1, k_2 \in \mathbb{Z}
\]
The oracle: \( \text{conv}(T) \), continued

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\lambda u_2 + v_2 = k_2 \det([u|v])
\end{array} \right., \ \lambda, k_1, k_2 \in \mathbb{Z}$$
The oracle: \( \text{interior}(L_\alpha) \)

2. Assuming \( \text{conv}(T) \) lattice-free,

Prop.: It is enough to check 2 or 3 specific integer points:
Solver tricks: callbacks

Solving slave MIPs

\[
\begin{align*}
\min & \quad \tilde{\alpha}^T x \\
\text{s.t.} & \quad x \subseteq P,
\end{align*}
\]

- Feasible solution \( \hat{x} \) with \( \tilde{\alpha}^T \hat{x} < \tilde{\alpha}_0 \)
  \( \rightarrow \hat{x} \) can be added to \( S \).

- Dual bound \( \bar{z} \) reaches \( \tilde{\alpha}_0 \),
  \( \rightarrow (\tilde{\alpha}, \tilde{\alpha}_0) \) is valid for \( P \).
Solver tricks: callbacks

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Two-row relaxation: which models?

- We are still far from a closure
  - What reasonable set of two-models can we select?
    - All models read from a simplex tableau
    - $O(m^2)$ two-row models
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“all” two-row models: separation loop

Let \( x^* \leftarrow \) LP optimum
Read the two-row models from optimal tableau.
Read and add GMIs from that tableau.

\[
\text{do } \{
\quad \text{Let } x^* \leftarrow \text{new LP optimum.}
\quad \text{Separate } x^* \text{ with the two-row models.}
\}\quad \text{while (cuts were found).}
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“all” two-row models: results

Computations on the 62 MIPLIB 3.0 (preprocessed) instances for which

(a). the integrality gap is not zero, and
(b). an optimal MIP solution is known.
“all” two-row models: results

We have a result for 55/62 instances (4 numerical, 3 memory).

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<tr>
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For 13 instances, the separation is exact.
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For 13 instances, the separation is exact.
Heuristic selection of two-row models

Issue:
- $O(m^2)$ is already a large number of models

Hypothesis:
- Not all models are necessary to achieve good separation

Rationale:
- MIPLIB models are mostly sparse
- Multi-cuts from rows with no common support are linear combinations of the corresponding one-row cuts
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▶ MIPLIB models are mostly sparse
▶ Multi-cuts from rows with no common support are linear combinations of the corresponding one-row cuts
Heuristic selection of two-row models: results

With an arbitrary limit of $m$ two-row models, we have a result for 58/62 instances (1 numerical, 3 memory).

On the 55 common results,

<table>
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<tr>
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With an arbitrary limit of \( m \) two-row models, we have a result for 58/62 instances (1 numerical, 3 memory).

On the 55 common results,

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facet-defining $\alpha^T x \geq \alpha_0$ \iff facet-defining $\alpha^T x + \alpha_0 x_0 \geq 0$
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Polarity for general polyhedra: Conify

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Conify: $P$ is a polytope

Note: $P = \text{proj}_x(P^+ \cap \{x_0 = -1\})$.

$P^+ = \{(x, x_0) \in \mathbb{R}^{n+1} : x_0 \leq 0, \ x \in -x_0P\}$
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$Q$ is the polar of $P^+$ ⇔ $P^+$ is the polar of $Q$

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facet-defining $\beta^T x \geq 0$ \iff extreme ray $\beta$

valid $\gamma^T x = 0$ \iff $\gamma$ in the linearity space
Going back to general (full-dimensional) polyhedra

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$$\alpha^T x \geq \alpha_0$$  \quad $$\alpha^T x + \alpha_0 x_0 \geq 0$$  \quad $$\mathrm{ray} \ (\alpha, \alpha_0)$$

$$\alpha^T x = \alpha_0$$  \quad $$\alpha^T x + \alpha_0 x_0 = 0$$  \quad $$(\alpha, \alpha_0) \ \mathrm{in \ lin.sp.}$$