Changes to RHS:

Consider

\[
\begin{align*}
\text{min } & \quad c^T x \\
\text{st. } & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]  

\[
\begin{align*}
\text{min } & \quad c^T x \\
\text{st. } & \quad A x = b + \theta \cdot e_i \\
& \quad x \geq 0
\end{align*}
\]  

for \( \theta \in \mathbb{R} \)

**Theorem:** Let \( B \) be optimal for \((P)\). \( B \) is optimal for \((P')\) if and only if

\[
\begin{align*}
B^{-1} b + \theta \cdot B^{-1} e_i & \geq 0 \\
\end{align*}
\]

From lecture 12
Proof. \( B \) is optimal for \((P)\)

\[
\Rightarrow \quad b = B^{-1}b' \geq 0
\]

\[
\bar{c}^T = c^T - c_{B}^T B^{-1}A \geq 0 \tag{2}
\]

\( B \) is optimal for \((P')\) iff

\[
\bar{b}' = B^{-1}b' = B^{-1}(b + \theta e_i)
\]

\[
\bar{c} = c^T - \bar{c}_B B^{-1}A = c^T - c_{B}^T B^{-1}A
\]

\[
\bar{c} \geq 0
\]

always holds, by (2)

[From lecture 12]
Observe that even if \( B \) stays optimal \((p')\),
\[
\tilde{x}_B' = B^{-1}b' = B^{-1}(b + \theta e_i) = \overline{B^{-1}b} + \theta B^{-1}e_i \neq \overline{x}_B
\]

However,
\[
\overline{y}' = (B^T)^{-1}c_B' = (B^T)^{-1}c_B = \overline{y}
\]

By strong duality \( z' = c^T\tilde{x}' = b^T\overline{y}' \)
\[
= (b + \theta e_i)^T\overline{y}
= b^T\overline{y} + \theta e_i^T\overline{y}
= z + \theta \overline{y};
\]

[from lecture 12]
Example: House Depot produces:

- hammers (fixed price 130)
  from 1.5 kg of steel, 1 rivet, 0.3 kg of plastic, and
- pliers (fixed price 100)
  from 1 kg of steel, 1 rivet, 0.5 kg of plastic.

Current stocks are:

- 27 kg of steel, 21 rivets, 9 kg of plastic.

Maximize income from current stocks.

(Assume that fractional hammers and pliers are ok)

[From lecture II]
VAR: \[ x_1: \text{ hammers} \]
\[ x_2: \text{ pliers} \]

MODEL: \[
\text{max} \quad 130 \cdot x_1 + 100 \cdot x_2 \]
\[
\text{s.t.} \quad 1.5 \cdot x_1 + 1 \cdot x_2 \leq 27 \quad \text{e-steel}
\]
\[
1 \cdot x_1 + 1 \cdot x_2 \leq 21 \quad \text{rivets}
\]
\[
0.3 \cdot x_1 + 0.5 \cdot x_2 \leq 9 \quad \text{e-plastic}
\]
\[
x_1, x_2 \geq 0
\]

[From lecture 11]
Example: hammers & pliers (cont'd)

S.E.F: \[
\begin{align*}
\text{min} & \quad \begin{bmatrix} -130 & -100 & 0 & 0 & 0 & 0 \end{bmatrix} x \\
\text{s.t.} & \quad \begin{bmatrix} 1.5 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0.3 & 0.5 & 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 27 \\
21 \\
9 \end{bmatrix} \\
x \geq 0
\end{align*}
\]

[simplex method =>]

Optimal basis: \( B = \{1, 2, 5\} \)

\( x^* = (12, 9, 0, 0, 0, 0.9) \)

\( y^* = (-60, -40, 0) \)

\( z^* = -2460 \leftarrow \text{-profit} \)
optimal tableau:

\[
\begin{bmatrix}
0 & 0 & 60 & 40 & 0 \\
1 & 0 & 2 & -2 & 0 \\
0 & 1 & -2 & 3 & 0 \\
0 & 0 & 0.4 & -0.9 & 1
\end{bmatrix} \begin{bmatrix} x \\ 12 \\ 9 \\ 0.9 \end{bmatrix} \geq 0
\]

optimal basis matrix:

\[
B = \begin{bmatrix}
1.5 & 1 & 0 \\
1 & 1 & 0 \\
0.3 & 0.5 & 1
\end{bmatrix}
\quad B^{-1} = \begin{bmatrix}
2 & -2 & 0 \\
-2 & 3 & 0 \\
0.4 & -0.9 & 1
\end{bmatrix}
\]
Q1: We change the available steel from 27 to 27 + \theta.
For what values of \theta does the basis remain optimal?

\[ b' = b + \theta \cdot e, \]

\[ = \begin{bmatrix} 27 \\ 21 \\ 9 \end{bmatrix} + \theta \begin{bmatrix} 1' \\ 0 \\ 0 \end{bmatrix} \]

\[ \overline{b'} = B'^{-1} \cdot b' = B'^{-1} \cdot (b + \theta e), \]

\[ = B'^{-1}b + \theta \overline{B'^{-1}e}, \]

(first column of \overline{B'^{-1}})
\[
\begin{bmatrix}
12 \\
9 \\
0.9
\end{bmatrix}
+ \theta 
\begin{bmatrix}
2 \\
-2 \\
0.4
\end{bmatrix} \geq 0
\]

\[
\begin{align*}
12 + \theta \cdot 2 & \geq 0 \\
9 + \theta \cdot (-2) & \geq 0 \\
0.9 + \theta \cdot 0.4 & \geq 0
\end{align*}
\]

\[\Rightarrow \begin{cases}
\theta \geq -6 \\
\theta \leq 4.5 \\
\theta \geq -2.25
\end{cases}\]

\[\Rightarrow \theta \in [-2.25, 4.5]\]
Definition: The set of all $\theta$ such that $B$ stays optimal is called the allowable range.
Q2. What is the allowable range for plastic?

\[ b' = \begin{bmatrix} \frac{27}{2} \\ \frac{9}{2} \\ 0 \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leq c_3 \]

\[ b' = \begin{bmatrix} 12 \\ 5 \\ 0.9 \end{bmatrix} + \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \geq 0 \]

third column of \( B' \)

always true

\( \begin{cases} 12 + 0. \theta & \geq 0 \\ 5 + 0. \theta & \geq 0 \iff \theta \geq -0.9 \\ 0.9 + 1. \theta & \geq 0 \end{cases} \)

\[ \theta \in [-0.9, +\infty] \]
Q3. A seller proposes 4 additional kg of steel for $200. Should you accept?

Note: from \( A_1, \theta = 4 \in [-2.25, 4.5] \), i.e. 4 is in allowable range for steel.

\[ 2^* = [-130 -100 0 0 0]. x^* = b^T. y^* \]

\[ = (b + \theta e,)^T y^* \]
\[ = b^T y^* + \theta e^T y^* \]
\[ = 2^* + \theta . y^* \]
\[ = -2460 - 60 \theta \]
\[ = -2460 - 240 \]

(\text{profit}) decreases by \text{240}

profit increases by \text{240}

\[ \rightarrow \text{ACCEPT!} \]
Q4 Someone is willing to buy 2 kg of steel off our stock for $130. Accept?

Note: \( \Theta = -2 \in [-2.25, 4.5] \)

- basis stays the same

\[
Z^* = Z^* + \Theta \cdot y^*
\]

\[
= -2460 + (-2) \cdot (-60)
\]

\[
= -2460 + 120
\]

(-profit) increases by 120

profit decreases by 120

\( 120 < 130 \)

\( \rightarrow \) ACCEPT!
Changes to objective function

Consider

\[ \min \ c^T x \]
\[ \text{s.t.} \quad A x = b \]
\[ x \geq 0 \]

\[ (p) \]

\[ \min \ (c + \theta \cdot e_j)^T x \]
\[ A x = b \]
\[ x \geq 0 \]

\[ (p') \]

for \( \theta \in \mathbb{R} \)
Theorem: Let \( B \) be optimal for \((P)\). \( B \) is optimal for \((P')\) if and only if

\[
\begin{align*}
\bar{c}_j + \theta &\geq 0 \quad \text{if } j \in B \\
\bar{c}_n - \theta e_i^T B^{-1} N &> 0 \\
\end{align*}
\]

if \( j \) is the \( i \)th basic column.

nonbasic part of \( i \)th row of optimal tableau.
Proof: $\bar{B}$ is optimal for $(P) \Rightarrow$

\[
\bar{b} = B^{-1}b = 0 \quad (1)
\]

\[
\bar{c}^T = c^T - c_B^T B^{-1}A \geq 0 \quad (2)
\]

$\bar{B}$ is optimal for $(P')$ iff

\[
\bar{b}' = B^{-1}b' = B^{-1}b = \bar{b} = 0
\]

always holds, by (1)

\[
\bar{c}' = c'^T - c_B'^T B^{-1}A \geq 0
\]
Case 1: \( j \neq B \)

\[
\overline{z}' = (z + \theta e_j)^T - c_B^T B^{-1} A
\]

\[
= c^T - c_B^T B^{-1} A + \theta e_j^T
\]

\[
\geq \frac{\overline{z}^T + \theta e_j^T}{\theta} \geq 0
\]

\( \iff \)

\[
\overline{z}_j + \theta \geq 0
\]
Case 2: \( j \in \mathcal{B} \), \( j \) is the \( i \)th basic column

\[
\bar{c}' = (c + \theta e_j)^T - (c_B + \theta e_i)^T \mathbf{B}'^{-1} \mathbf{A} \\
= \bar{c}^T - c_B^T \mathbf{B}'^{-1} \mathbf{A} + \theta e_j^T - \theta e_i^T \mathbf{B}'^{-1} \mathbf{A} \\
= \bar{c}^T - \theta (e_i^T \mathbf{B}'^{-1} \mathbf{A} - e_j^T) \geq 0
\]

(\( i \)th row of optimal tableau)

(\( i \)th row of optimal tableau)

With a zero for basic variable

\[
\Leftarrow \quad \bar{c}_N^T = \theta \cdot e_i^T \mathbf{B}'^{-1} \mathbf{N} \geq 0
\]
What is the allowable range for $c_i$?

Note: $i \in B$, first basic variable

$$
\bar{c}_i^T = c_i^T - c_B^T B^{-1} \mathbf{A}
= (c_i^T + \theta e_i^T) - (c_B^T + \theta e_i^T) B^{-1} \mathbf{A}
= c_i^T - c_B^T B^{-1} \mathbf{A} - \theta (e_i^T B^{-1} \mathbf{A} - e_i)
= \bar{c}_i^T - \Theta[[1 \ 0 \ 2 \ -2 \ 0] - [X \ 0 \ 0 \ 0 \ 0]]
= [0 \ 0 \ 60 \ 40 \ 0] - \Theta[0 \ 0 \ 2 \ -2 \ 0]
\geq 0
$$
\[
\begin{align*}
&\iff
\\&\begin{cases}
0 - 0.0 \theta &\geq 0 \\
0 - 0.0 \theta &\geq 0 \\
60 - 2.0 \theta &\geq 0 \\
40 + 2.0 \theta &\geq 0 \\
0 - 0.0 \theta &\geq 0
\end{cases}
\\&\iff \\
&\begin{cases}
\theta \leq 30 \\
\theta \geq -20
\end{cases}
\iff \theta \in [-20, 30]
\end{align*}
\]

**Warning:** in terms of original costs (max), we have 
\[-\theta \in [-30, 20]\]