Bases & duality

Consider the primal-dual pair:

\[
\begin{align*}
\min \ & c^T x \\
\text{st.} \ & Ax = b \\
& x \geq 0 \\
\end{align*}
\] \hspace{1cm}
\[
\begin{align*}
\max \ & b^T y \\
\text{st.} \ & A^T y \leq c \\
& y \text{ free}
\end{align*}
\] \hspace{1cm} (P) \hspace{1cm} (D)

Observe that \((P)\) is in S.E.F., so we have a concept of a basis for \((P)\). \((D)\) is not in S.E.F.

Q: Can a basis \(B\) of \((P)\) tell us something about \((D)\)?
Intuition: Let $B$ be a basis for $(P)$.

\[
\begin{align*}
\min & \quad \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} x \\
\text{s.t.} & \quad \begin{bmatrix} B & N \end{bmatrix} x = b \quad (P) \\
\end{align*}
\]

\[
\begin{align*}
\max & \quad b^T y \\
\text{s.t.} & \quad \begin{bmatrix} B^T & N^T \end{bmatrix} y \leq \begin{bmatrix} c_B^T \\ c_N^T \end{bmatrix} \\
& \quad x \geq 0
\end{align*}
\]

Let $x^*$ be optimal for $(P)$, with $x^* = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$, free

let $y^*$ be optimal for $(N)$.

We know that $x^*_B \geq 0$, assume $x^*_B > 0$.

Then, by complementary slackness, the corresponding constraints in $(N)$ are tight: $B^T y^* = c_B$

Therefore, $y^* = (B^T)^{-1} . c_B$
Definitions:

- \( \bar{x} \) is the (primal) basic solution associated to \( \mathcal{B} \) if \( \bar{x}_N = 0 \) and \( \bar{x}_\mathcal{B} = \mathcal{B}^{-1} b \)

- \( \bar{y} \) is the dual basic solution associated to \( \mathcal{B} \) if \( \bar{y} = (\mathcal{B}^T)^{-1} c_\mathcal{B} \).

- \( \mathcal{B} \) is (primal) feasible if \( \bar{x} \) is feasible for (\( \mathcal{P} \))

- \( \mathcal{B} \) is dual feasible if \( \bar{y} \) is feasible for (\( \mathcal{D} \)).
Theorem: Let $B$ be a basis associated with primal solution $\bar{x}$ and dual solution $\bar{y}$. The following are equivalent:

1) $\bar{x}$ feasible for (P) and $\bar{y}$ feasible for (D)
2) $\bar{x}$ is optimal for (P)
3) $\bar{y}$ is optimal for (D)
Proof:
(1) \text{ or } (2): \text{ Recall that } \bar{y} = (B^T)^{-1} \zeta_B

\text{Observe that: } \bar{y} \text{ feasible } \iff A^T \bar{y} \leq c

\iff A^T (B^T)^{-1} \zeta_B \leq c

\iff c - A^T (B^T)^{-1} \zeta_B \geq 0

\iff c^T - \zeta_B^T B^{-1} A \geq 0

\iff \bar{c} \geq 0

\{ \bar{y} \text{ feasible } \} \iff \{ \bar{x} \text{ feasible } \} \iff c \geq 0 \iff \bar{x} \text{ optimal}
(1) => (3):

Observe that:

\[ c^T \bar{x} = c_{\beta}^T \bar{x}_{\beta} = c_{\beta}^T \beta^{-1} b \]

\[ b^T \bar{y} = \bar{y}^T b = \left( (\beta^{-1})^T c_{\beta} \right)^T b = c_{\beta}^T \beta^{-1} b \]

\[ \left\{ \begin{array}{l}
    c^T \bar{x} = b^T \bar{y} \\
    \bar{x} \text{ feasible} \\
    \bar{y} \text{ feasible}
\end{array} \right\} \iff \left\{ \begin{array}{l}
    \bar{x} \text{ optimal} \\
    \bar{y} \text{ optimal},
\end{array} \right\} \]

by strong duality.

(3) => (1):

put (0) in SEF, then similar to (2) => (1).
**Definition:** Let $B$ be a basis associated with primal basic solution $\bar{x}$ and dual basic solution $\bar{y}$.

$B$ is **optimal** if $\bar{x}$ is optimal (or, equivalently, if $\bar{y}$ is optimal).
Sensitivity analysis

Consider $\min c^T x$
\[s.t. \ A x = b \quad x \geq 0\]

$(P)$

Suppose that $B$ is an optimal basis for $(P)$.

Sensitivity analysis answers the following question:

For what changes of $A, b, c$ does $B$ still give an optimal solution?
changes to RHS:

Consider

\[ \begin{align*}
\text{min} & \quad c^T x \\
\text{st.} & \quad A x = b \\
& \quad x \geq 0
\end{align*} \]

\[ (P) \]

\[ \begin{align*}
\text{min} & \quad c^T x \\
\text{st.} & \quad A x = b + \theta e_i \\
& \quad x \geq 0
\end{align*} \]

\[ (P') \]

for \( \theta \in \mathbb{R} \)

Theorem: Let \( B \) be optimal for \( (P) \). \( B \) is optimal for \( (P') \) if and only if

\[ \frac{B^{-1} b + \theta B^{-1} e_i}{b} \geq 0 \]

\( \text{for the } i^{th} \text{ column of } B^{-1} \)
proof. \( B \) is optimal for \((P)\)
\[
\bar{b} = B^{-1}b \geq 0 \quad (1)
\]
\[
\bar{c}^T = c^T - c_{\beta}^T B^{-1}A \geq 0 \quad (2)
\]

\( B \) is optimal for \((P')\) iff
\[
\bar{b}' = B^{-1}b' = B^{-1}(b + \theta \varepsilon_i)
\]
\[
= B^{-1}b + \theta B^{-1}e_i \geq 0
\]
\[
\bar{c} = c^T - c_{\beta}^T B^{-1}A = c^T - c_{\beta}^T B^{-1}A
\]
\[
\bar{c} \geq 0
\]
always holds, by \((2)\)
Observe that even if \( P \) stays optimal \((p')\),

\[
\bar{x}'_3 = \bar{B}'_3 b'_3 = \bar{B}'_3 (b + \theta e_i) = \bar{B}'_3 b + \theta \bar{B}'_3 e_i \neq \bar{x}_3
\]

However,

\[
\bar{y}'_3 = (\bar{B}^T)^{-1} c'_3 = (\bar{B}^T)^{-1} c_{33} = \bar{y}
\]

By strong duality, \( z' = c^T \bar{x}' = b^T \bar{y}' \)

\[
= (b + \theta e_i)^T \bar{y}
= b^T \bar{y} + \theta e_i^T \bar{y}
= z + \theta \bar{y};
\]