Previous lecture

Consider

\[
\begin{align*}
\min & \quad \bar{c}^T x \\
\text{s.t.} & \quad A x = b \\
& \quad x \geq 0.
\end{align*}
\] (P)

Given a basis \( B \), (P) is equivalent to

\[
\begin{align*}
\min & \quad \bar{c}^T x \\
\text{s.t.} & \quad \bar{A} x = \bar{b} \\
& \quad x \geq 0,
\end{align*}
\] (tableau)

where

\[
\begin{align*}
B &= A_B \\
\bar{c}^T &= c^T - c_B^T B^{-1} A \\
\bar{A} &= B^{-1} A \\
\bar{b} &= B^{-1} b
\end{align*}
\]
properties of tableau:

\[
\begin{align*}
\mathbf{c}^T &= \begin{bmatrix} 0^T & 1 \end{bmatrix} \\
\mathbf{A} &= \begin{bmatrix} I & \mathbf{A}_N \\
\end{bmatrix}
\end{align*}
\]

\[
\begin{array}{c}
\mathbf{B} \\
\end{array}
\]

corresponding basic solution:

\[
\begin{align*}
\mathbf{x} &= 0 \\
\mathbf{A}_N \mathbf{x} &= \begin{bmatrix} I & \mathbf{A}_N \end{bmatrix} \begin{bmatrix} \frac{x_B}{x_N} \end{bmatrix} = \mathbf{b}
\end{align*}
\]

yields

\[
\begin{align*}
I \mathbf{x}_B &= \mathbf{b}
\end{align*}
\]

i.e.,

\[
\mathbf{x} = \begin{bmatrix} \mathbf{b} \\
\mathbf{0}
\end{bmatrix}
\]

note: basic feasible solutions \(\iff\) vertices of \(\mathbf{P}\)
We want to go from a feasible basis to a better feasible basis (without enumerating all bases)

**Def.** Let \( l \in B \), \( e \in N \).

A pivot is the action of creating

\[ B' = B \setminus \{el\} \cup \{e\}. \]

\( x_e \) is the entering variable.

\( x_l \) is the leaving variable.
If we pivot away from current basis, 

- one nonbasic variable, \( x_e \) (currently = 0) will become basic (hence \( \geq 0 \)).
- the value of the basic variables will change to preserve feasibility (\( \geq 0 \)).
- how will the objective value change?

- \( \forall j \in B, \bar{c}_j = 0 \) so \( x_j \) has no impact.
- \( \forall j \in N \setminus \{e\}, \bar{x}_j = 0 \) and stays zero so no impact.
- \( x_e \) can increase, so objective can decrease if \( \bar{c}_e < 0 \).
Example tableau

\[ \text{min} \]

\[ 3x_4 - 2x_5 \]

\[ x_1 + 2x_4 - x_5 = 2 \]

\[ x_2 - x_4 + x_5 = 1 \]

\[ x_3 + 2x_5 = 1 \]

\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \]

basic solution: \( \bar{x} = (2, 1, 1, 0, 0) \)
In the example,

\[ x_5 \] enters the basis \( \Rightarrow \)

(all other nonbasics, i.e. \( x_4 \), are fixed to 0)

\[
\begin{align*}
  x_1 &= 2 + x_5 \geq 0 \\
  x_2 &= 1 - x_5 \geq 0 \quad \Rightarrow \quad 0 < x_5 \leq 1 \\
  x_3 &= 1 - 2x_5 \geq 0 \quad \Rightarrow \quad -2x_5 \leq 1
\end{align*}
\]

\( \Rightarrow \quad x_5 \) will increase to \( \frac{1}{2} \)

\( x_3 \) become zero, can leave

the basis
When $x_e$ increases:

if $\bar{a}_{ie} \leq 0$, $x_j$ increases or stays unchanged.

if $\bar{a}_{ie} > 0$, $x_j$ decreases:

\[
\text{when } x_e \text{ reaches } \frac{\bar{b}_i}{\bar{a}_{ie}}, \text{ } x_j \text{ reaches } 0
\]

$\Rightarrow$ Ratio test:

by how much can $x_e$ increase?

\[
\lambda = \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{ie}} \mid \bar{a}_{ie} > 0 \right\}
\]

Leaving variable: any $i$-th basic variable such that $\bar{a}_{ie} > 0$ and $\lambda = \frac{\bar{b}_i}{\bar{a}_{ie}}$. 
More formally,

Let $x_e$ enter the basis (with $\bar{c}_e < 0$).

Consider $x_j$ where $j \in B$, $j$ is the $i$-th basic variable.

$$
\bar{b}_i - \overline{\bar{a}_{ie}} \cdot x_e = \bar{b}_i
$$

So $x_j = \bar{b}_i - \overline{\bar{a}_{ie}} \cdot x_e$
Simplex method

Given a basis $B$ such that $\overline{b} \geq 0$,

- choose entering variable $x_e$: $\overline{c}_e < 0$

- choose pivot row

$$i = \text{argmin}_i \left\{ \frac{\overline{b}_i}{\overline{a}_{i e}} \mid \overline{a}_{i e} > 0 \right\}$$

- leaving variable $x_i$ is the basic variable in row $i$.

- pivot ($x_e$ enters, $x_i$ leaves)
The algorithm ends when
- there is no entering variable (optimality)
- there is no leaving variable (unboundedness)
**Duality**

Consider: \[ z = \max 2x_1 + x_2 \]

s.t. \[ x_1 + 2x_2 \leq 2 \]
\[ x_1 + x_2 \leq 2 \]
\[ x_1 - x_2 \leq 0.5 \]

Consider: \( \bar{x} = (1, 0.5), \quad \bar{z} = 2.5 \)

**Q:** Is \( \bar{x} \) optimal?

**A:** It is optimal if \[ 2x_1 + x_2 \leq 2.5 \] for all feasible \( x \).

**Q:** Could we show \[ 2x_1 + x_2 \leq U \] for some \( U \)?

**A:** Take linear combinations of constraints (with \( \geq 0 \) coefficients).
\[
\begin{align*}
  x_1 + 2x_2 & \leq 2 \quad (x \frac{1}{3}) \\
  x_1 + x_2 & \leq 2 \quad (x \frac{1}{3}) \\
  x_1 - x_2 & \leq 0.5 \quad (x \frac{2}{3}) \\
 \hline
  2x_1 + x_2 & \leq 3 \\
\end{align*}
\]

(still not sure if \( x \) is optimal)
General approach:

\[ x_1 + 2x_2 \leq 2 \]
\[ x_1 + x_2 \leq 2 \]
\[ x_1 - x_2 \leq 0.5 \]

\[ (y_1 + y_2 + y_3) x_1 + (2y_1 + y_2 - y_3) x_2 \leq 2y_1 + 2y_2 + 0.5y_3 \]

We want:
\[ 2x_1 + x_2 \leq 2y_1 + 2y_2 + 0.5y_3 \]

It will work as long as:
\[ y_1 + y_2 + y_3 = 2 \]
\[ 2y_1 + y_2 - y_3 = 1 \]
\[ y_1, y_2, y_3 \geq 0 \]

In which case it gives
\[ U = 2y_1 + 2y_2 + 0.5y_3 \]
We want the strongest possible upper bound $U$.

$$U = \min \ 2y_1 + 2y_2 + 0.5y_3$$

$$y_1 + y_2 + y_3 = 2$$

$$2y_1 + y_2 - y_3 = 1$$

$$y_1, y_2, y_3 \geq 0$$

(D) is the dual of (P)

Note: $y^* = (1, 0, 1)$ feasible for (D) gives $U = 2.5 \Rightarrow x$ optimal for (P)