Previous Lecture

Max s-t flow formulation:

\[(P) \quad \max \sum_{e \in S^+(q)} x_e - \sum_{e \in S^-(q)} x_e\]

where \(f_x(q) = 0\) for all \(q \in V \setminus \{s, t\}\) \((u \geq 0)\)

Property: If \(u \in \mathbb{Z}^n\), then \(x^*\) optimal for \((P)\) and \(x^* \in \mathbb{Z}^n\).
Application of Property: Bipartite Matchings

Let $G = (V, E)$ be an (undirected) graph.

**Def.** $M \subseteq E$ is a matching if no two edges of $M$ are incident to a same vertex.

[Diagram of a graph with matching edges marked in blue and non-matching edges marked in purple]
Def. $G$ is bipartite if there exists a partition $A, B$ of $V$, i.e.
. $A \cap B = \emptyset$
. $A \cup B = V$
such that every edge has one end in $A$ and the other in $B$. 
Problem: Find a maximum-cardinality matching \( M \) in a bipartite graph \( G \).

Naive formulation

**VAR:** \( x_e = \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{otherwise} \end{cases}, \forall e \in E \)**

**MODEL:**

\[
\begin{align*}
& \text{max} \sum_{e \in E} x_e \\
& \text{s.t. } \sum_{e \in \delta^{-}(v)} x_e \leq 1, \quad \forall v \in V \\
& \quad \left(x_e \in \{0, 1\}\right) \\
& \quad 0 \leq x_e \leq 1, \quad \forall e \in E \\
& \quad x_e \in \mathbb{Z}, \quad \forall e \in E
\end{align*}
\]
Flow formulation

Assume $E = \{ qr : q \in A, r \in B \}$

Let $G' = (V', E')$, where

- $V' = V \cup \{ s, t \}$
- $E' = E \cup \{ sq : Aq \in A \} \cup \{ qt : Aq \in B \}$
VAR: \( x_e \) = flow on edge \( e \in E' \)

MAX-FLOW MODEL:

\[\begin{align*}
\text{max} & \quad f_x(s) \\
\text{s.t.} & \quad f_x(q) = 0 \quad \forall q \in V \setminus \{s,t\} \\
& \quad 0 \leq x_e \leq 1 \quad \forall e \in E'
\end{align*}\]

- \( f_x(s) \) = flow from \( A \) to \( B = \sum_{e \in E} x_e \)
- \( \forall q \in A, \quad f_x(q) = \sum_{e \in \delta_q^+(q)} x_e - \sum_{e \in \delta_q^-(q)} x_e \)

\[= \sum_{e \in \delta_q^-(q)} x_e - x_{sq} = 0 \quad x_{sq} \leq 1 \implies \sum_{e \in \delta_q^-(q)} x_e \leq 1 \]

So \((a')\) is equivalent to \((a)\) without \( x_e \in \mathbb{Z} \)
However, using the property of max-flow
exists an optimal solution \((\mathbf{x}')\) where \(x \in \mathbb{Z}_+,\)
\(\forall e \in E.\)
THE MIN-COST FLOW MODEL

DATA:
① $G = (V, E)$ directed graph
② capacity $u_e \geq 0 \quad \forall e \in E$
③ unit cost $c_e \in \mathbb{R} \quad \forall e \in E$
④ net flow $b_q \in \mathbb{R}^+$ "pushed" (or "pulled" if $b_q < 0$) at $q$, $\forall q \in V$

PROBLEM Minimize total cost
VAR: $X_e = \text{flow through arc } e \in E$

MODEL:

$\min \sum_{e \in E} c_e X_e$

s.t. $f_x(q) = b_q \quad \forall q \in V$

$0 \leq X_e \leq U_e \quad \forall e \in E$
Application of Min-cost Flow Model: Transshipment Problem

Consider a road network. The arcs are roads with given capacities, and given transportation costs. The vertices are either:

- storage centers that must send an amount $b_q > 0$ of goods, or
- retail centers that must receive an amount $b_q < 0$ of goods, or
- Transshipment centers ($b_q = 0$).

Minimize the cost of sending the goods.
Property: If $u \in \mathbb{Z}^n$ and $b \in \mathbb{Z}^m$, then $J_x^*$ optimal for $(T)$ such that $x^* \in \mathbb{Z}^n$. 
A bus of capacity $p$ drives through cities $1, \ldots, n$.
For every city $i < j$,

$$d_{ij} = \text{demand to go from } i \text{ to } j$$
$$f_{ij} = \text{fare to go from } i \text{ to } j$$

Objective: decide how many passengers to carry for every $i \rightarrow j$ trip,
maximize fares collected.