DEF. A directed graph $G$ is a pair $(V, E)$, where

. $V$ is a finite set (vertices)
. $E$ is a set of ordered pairs (arcs) of elements of $V$

Example 1:

$$V = \{s, a, b, t\}$$
$$E = \{sa, sb, ab, ba, at, bt\}$$
Notation:

Let $e = uv \in E$ \quad (u \rightarrow v)

- $e$ leaves $u$
- $e$ arrives at $v$

For any $q \in V$

$\delta^+(q) = \{ e \in E : e$ leaves $q \}$

$\delta^-(q) = \{ e \in E : e$ arrives at $q \}$
$\delta^+(a) = \{ at, ab \}$
$\delta^-(a) = \{ ba, sa \}$
MAX S-T FLOW PROBLEM

Given
1. $G = (V, E)$ directed graph
2. "source" $s \in V$
3. "sink" $t \in V$, $t \neq s$
4. "capacity" $u_e$, for every $e \in E$

We want that for every $q \in V \setminus \{s, t\}$, the "flow" entering $q = "flow"$ leaving $q$.
The "flow" on $e \in E$ is at most $u_e$.
Find maximum "flow" from $s$ to $t$. 
VAR: \( x_e \): amount of flow on arc \( ee \in E \).

Notation: for \( q \in V \),

\[
f_x(q) = \sum_{e \in S^+(q)} x_e - \sum_{e \in S^-(q)} x_e
\]

Max s.t. Flow Formulation:

\[
\max \quad f_x(s)
\]

\[
f_x(q) = 0 \quad \forall q \in V \setminus \{s, t\}
\]

\[
0 \leq x \leq u_e \quad \forall e \in E
\]
Example 1

Capacity U

Feasible flow \( x_e \)
Property of max s-t flow problems

\[
\begin{align*}
\max & \quad f_x(s) \\
\text{st.} & \quad f_x(q) = 0 \quad (q \neq s, t) \quad (P) \\
0 & \leq x_e \leq u_e
\end{align*}
\]

If \( u_e \) is integer for all \( e \), then among all the optimal solutions to \((P)\), at least one is integral.
**Note**: if $U_e$ is not integer:

$$S \rightarrow t \quad U_{ct} = 1.5$$

- There can still be other optimal solutions that are not integral.
Capacities

Optimal Solution