Solutions

Question 1  [3 marks] Circle the correct answer. No justification necessary.

1. In Julia/JuMP, the expression
\[
\sum_{i=1}^{5} 3i
\]

is written
(a) \text{sum}(i = 1:5, 3 * i)
(b) (correct) \text{sum}(3 * i \text{ for } i = 1:5)
(c) \text{sum}(3 * i \text{ for } i = [1, 5])
(d) \text{sum}(i = [1, 5], 3 * i)

2. In Julia/JuMP, the constraint
\[x + y \leq 5\]
can be written (for a model \text{md} with variables \text{x} and \text{y})
(a) (correct) @constraint(\text{md}, x + y <= 5)
(b) @constraint(\text{md}, x + y, <=, 5)
(c) @constraint(\text{md}, x + y) <= 5
(d) @constraint(\text{md}, \text{sum}(x, y)) <= 5

3. In Julia/JuMP, if a variable was declared using \texttt{@variable(md, x)} for a model \text{md}, its value can be accessed by writing
(a) \&x
(b) \text{x.value}
(c) \text{x.value()}
(d) (correct) \text{value(x)}
Question 2  [6 marks] Consider the following linear programming problem

\[
\min \begin{array}{c}
3x_2 + 2x_3 + x_4 \\
\end{array}
\begin{array}{c}
s.t. \\
\end{array}
\begin{array}{c}
x_2 + x_3 \geq 4 \\
x_1 + x_2 - x_3 + ax_4 \geq 4 \\
2x_2 - x_3 \geq 8 \\
x_1 - x_2 \geq 4 \\
x_3 + bx_4 \geq 3 \\
x_1, x_2, x_3, x_4 \in \mathbb{R}
\end{array}
\]

(P)

where \(a, b \in \mathbb{R}\) are constants. We do not know the value of \(a\) and \(b\), but we are told that \(y^* = \left(2, 0, 0.5, 0, 0.5\right)^T\) is a dual optimal solution. Let \(x^*\) be a primal optimal solution.

1. Write the dual of (P).

2. Find the numerical value of \(x^*_2\) and \(x^*_3\).

3. Find the numerical value of the objective function at \(x^*\).

4. Find the numerical value of \(x^*_4\).

5. Find the numerical value of \(b\).

Note that \(a\) and \(b\) can appear in the answer to (1), but the answers to (2)-(5) must be numerical.

Solution: 1. The dual of (P) is

\[
\max \begin{array}{c}
4y_1 + 4y_2 + 8y_3 + 4y_4 + 3y_5 \\
\end{array}
\begin{array}{c}
s.t. \\
\end{array}
\begin{array}{c}
y_2 + y_4 = 0 \\
y_1 + y_2 + 2y_3 - y_4 = 3 \\
y_1 - y_2 - y_3 + y_5 = 2 \\
ay_2 + by_5 = 1 \\
y_1, y_2, y_3, y_4, y_5 \geq 0
\end{array}
\]

(D)

2. By complementary slackness, the 1st, 3rd and 5th constraints of (P) are tight. In particular, the 1st and 3rd give

\[
\begin{cases}
x^*_2 + x^*_3 = 4 \\
2x^*_2 - x^*_3 = 8 \iff \begin{cases} x^*_2 = 4 \\
x^*_3 = 0
\end{cases}
\end{cases}
\]

3. The objective function in (D) gives us

\[z^* = 4 \cdot 2 + 4 \cdot 0 + 8 \cdot 0.5 + 4 \cdot 0 + 3 \cdot 0.5 = 13.5.\]

By strong duality, \(z^*\) is also the optimal objective function value for (P). Using the objective function in (P), we obtain that

\[z^* = 13.5 = 2 \cdot 4 + 2 \cdot 0 + x^*_4 \Rightarrow x^*_4 = 1.5\]

4. Using the fact that the 5th constraint is tight, we get

\[x^*_3 + bx^*_4 = 3 \iff 0 + b \cdot 1.5 = 3 \iff b = 2.\]
3. and 4. (alternative approach) We compute $b$ first. The last constraint of (D) is $ay_2 + by_5 = 1$. We know that the given dual solution $y^* = \begin{pmatrix} 2, & 0, & 0.5, & 0, & 0.5 \end{pmatrix}^T$ satisfies this constraint, yielding $a \cdot 0 + b \cdot 0.5 = 1$ hence $b = 2$.

Now, using the fact that the 5th constraint is tight, we get

$$x_3^* + bx_4^* = 3 \iff 0 + 2 \cdot x_4^* = 3 \iff x_4^* = 1.5$$

Question 3 [6 marks] Consider a set of $N$ objects indexed $\{1, \ldots, N\}$. For each pair $\{i, j\}$ of objects, we are given a constant $D_{ij} \in \mathbb{R}$, which measures the “affinity” between the objects. For all $i, j \in \{1, \ldots, N\}$, we have:

(i) $D_{ij} = D_{ji}$,
(ii) $D_{ii} = 0$,
(iii) $D_{ij}$ can be positive, negative or zero.

We want to select a subset of $K$ objects (out of the $N$ objects) that maximizes the sum of the pairwise affinities $D_{ij}$ between the selected objects.

For illustrative purposes, consider an example, with $N = 4$, $K = 3$, and

$$D = \begin{pmatrix} 0 & 3 & 2 & 2 \\ 3 & 0 & -1 & 1 \\ 2 & -1 & 0 & 4 \\ 2 & 1 & 4 & 0 \end{pmatrix}.$$ 

A subset $\{1, 2, 3\}$ would have a sum $D_{12} + D_{13} + D_{23} = 3 + 2 + (-1) = 4$. However, a subset $\{1, 3, 4\}$ would be better, with a sum $D_{13} + D_{14} + D_{34} = 2 + 2 + 4 = 8$.

Given constants $N > 0$, $K > 0$ and $D_{ij} \in \mathbb{R}$ for all $i, j \in \{1, \ldots, N\}$, model this problem as an integer programming problem. (Remark: your IP must be valid for all $N$, $K$ and $D$, not just for the example above, which is given only to illustrate the problem statement.)

Solution:

Variables:

$$x_i = \begin{cases} 1 & \text{if object } i \text{ is taken,} \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i = 1, \ldots, N.$$ 

$$z_{ij} = \begin{cases} 1 & \text{if objects } i \text{ and } j \text{ are both taken,} \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i = 1, \ldots, N, \quad j = 1, \ldots, i - 1$$ 

Model:

$$\max \sum_{i=1}^{N} \sum_{j=1}^{i-1} d_{ij} z_{ij}$$ 

s.t. 

$$\sum_{i=1}^{N} x_i = K$$ 

$$x_i + x_j \leq z_{ij} + 1 \quad \forall i = 1, \ldots, N, \forall j = 1, \ldots, i - 1$$ 

$$x_i \geq z_{ij} \quad \forall i = 1, \ldots, N, \forall j = 1, \ldots, i - 1$$ 

$$x_j \geq z_{ij} \quad \forall i = 1, \ldots, N, \forall j = 1, \ldots, i - 1$$ 

$$x_j \in \{0, 1\} \quad \forall i = 1, \ldots, N$$ 

$$z_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, N, \forall j = 1, \ldots, i - 1$$
Explanation: Regarding the objective function, we maximize the sum of the affinities. Note that we sum over all \(i, j\) such that \(j < i\), because we defined \(z_{ij}\) for those indices only. We could have defined \(z_{ij}\) for all \(i, j\), in which case we would have had to simply divide the corresponding sum by two.

The first constraint \(\sum_{i=1}^{N} x_i = K\) ensures that we select exactly \(K\) objects. The rest of the constraints ensure that \(z_{ij}\) is consistent with the value of \(x_i\). Specifically, for every \(z_{ij}\), we must force its value to 1 if both \(x_i\) and \(x_j\) are 1:

\[
z_{ij} \geq x_i + x_j - 1
\]  

(1)

and we force its value to 0 otherwise:

\[
\begin{align*}
  z_{ij} &\leq x_i , \\
  z_{ij} &\leq x_j
\end{align*}
\]  

(2)

or as an alternative

\[
z_{ij} \leq \frac{1}{2}(x_i + x_j).
\]

Remark that all the above constraints are necessary because, since \(d_{ij}\) can be of any sign, we don’t know if individual \(z_{ij}\) are minimized or maximized. Also, while there is (at least) one alternative way to implement (2), \(z_{ij} \geq \frac{1}{2}(x_i + x_j)\) is not a valid alternative to (1), since having \(x_i = 1\) and \(x_j = 0\) would already force \(z_{ij} = 1\).