PMath 360 — Sample Final — 2003

1. **Collineation**
   Let four points in the Cartesian plane be given:

$$A:\ (0,0),\ B:\ (1,0),\ C:\ \left(\frac{9}{10},\frac{7}{10}\right),\ \text{ and } D:\ \left(0,\frac{8}{10}\right).$$

   (a) Find the homogeneous coordinates of A, B, C, D and the two points $AB \cap CD$ and $BC \cap DA$.

   (b) Find the 3 by 3 matrix of the collineation that maps the homogeneous coordinates of the four points of the unit square A, B, $(1,1)$ and $(0,1)$ to the homogeneous coordinates of A, B, C and D respectively. (*Hint:* Use the answers to (a).)

2. **Straight edge and compass constructions**

   (a) Give the best straight edge and compass construction you know for the inverse of a point P with respect to a circle C.

   (b) Give a straight edge and compass construction for the circle through three points P, Q and R (not collinear).

3. **Orthogonal circle construction**
   You are given two circles C_1 and C_2 and a point X not on either circle. You are also given that X is not on the line of centres of C_1 and C_2. Present a step by step construction for the circle through X that is orthogonal to both C_1 and C_2. (You may use the results of question 2, if you wish to.)

4. **Hermitian matrices**
   Consider the pencil of circles formed by the circles given by the two Hermitian matrices

$$H_1 = \begin{bmatrix} 1 & -1+2i \\ -1-2i & 4 \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} 1 & -3+2i \\ -3-2i & 4 \end{bmatrix}.$$

   Calculate:

   (a) $\triangle_1$

   (b) $\triangle_2$

   (c) $\triangle_{12}$

   (d) the cosine of the angle between the two circles.

5. **Stereographic projection**

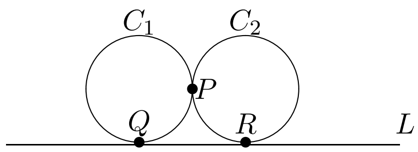
   Consider the sphere $\mathcal{S}$ with centre $(0, 0, 0)$ and radius 1 and the stereographic projection from $\mathcal{S}$ to the complex plane, as developed in class, given by this correspondence $(u, v, w) \leftrightarrow \dfrac{u + iv}{1 + w}$.

   (a) **From the plane to the sphere**
   A circle in the complex plane is given with centre $2 + 3i$ and radius 4. Find the plane that contains the circle on the sphere that corresponds to this circle.

   (b) **From the sphere to the plane**
   A certain circle on the sphere lies in the plane $2u + 4v + 3w = 1$. Find the circle in the complex plane that corresponds to this.

6. **Tangent circle**

   Two circles with the same radius r are tangent to each other at a point P. They are also tangent to a common line L at Q and R as in the figure below. Give the steps that will allow you to construct the circle in the region bounded by these three objects that is tanget to all three objects.



7. **Orthogonal Circles**

   Let C_1 and C_2 be any two circles with centres O_1 and O_2, respectively. Let X be any point not on the line of centres. In question 3, a circle was created to pass through X and to be orthogonal to both C_1 and C_2. Call that circle $K(X)$.

   Suppose further that C_1 and C_2 are disjoint and that $K(X)$ meets O_1O_2, the line of centres, in two points. Let Z be one of the intersection points. Let Σ be any circle whose centre is Z. Let C_1' and C_2' be the inverses with respect to Σ of C_1 and C_2, respectively. Prove that C_1' and C_2' are concentric.

8. **Poincaré model of the Hyperbolic plane**

   (a) Given A and B, two $P-$points interior to a base circle Ω. State the steps in the construction for the $P-$line on A and B.

   (b) Given two P-points A and B and the P-line L_1 on A and B, find the P-line L_2 that reflects (inverts) A into B.