PMath 360 Sample Final 2002

1. Warmup. Given five points $A:(1,-5,2)$, $B:(1,2,-1)$, $C:(2,-1,0)$, $D:(5,1,-1)$ and $F:(-12,1,1)$ follow the steps below to find the coordinates for the line $(AB \cap CD)F$.

[2] (a) $L_1 = \text{line}(A, B)$

[2] (b) $L_2 = \text{line}(C, D)$

[2] (c) $E = \text{point}(L_1, L_2)$

[4] (d) $L_3 = \text{line}(E, F)$

2. Find the matrix of the collineation that maps the standard frame to

[5] (a) $(0,1,1)$, $(2,1,0)$, $(1,0,3)$, $(-4,1,10)$.

[5] (b) Use (a) or otherwise to find the matrix of the collineation that maps the frame above to the standard frame.

3. Seven points are given: $V:(1,1,1)$, $A:(1,0,0)$, $B:(0,1,0)$, $C:(0,0,1)$
$A':(1+a,1,1)$, $B':(1,1+b,1)$, $C':(1,1,1+c)$.

[4] (a) Prove that the two triangles ABC and $A'B'C'$ are in perspective from a point.

[6] (b) Prove Desagues Theorem by showing directly that triangles ABC and $A'B'C'$ are perspective from a line.

[10]4. Find the matrix of the conic that satisfies these conditions:

(a) The polar of $(1,0,0)$ is $[0,1,3]$.

(b) The polar of $(-2,1,0)$ is $[1,2,-5]$.

(c) The point $(4,1,1)$ is on the conic.

[10]5. The following construction is given:

$\mathcal{C}$ is a circle with center O;

P is a point interior to $\mathcal{C}$;

$\ell_1 = \text{line } (O, P)$;

$\ell_2 = \text{perpendicular } (P, \ell_1)$;

$\{S, T\} = \text{meet } (\mathcal{C}, \ell_2)$;

$\ell_3 = \text{line } (O, S)$;

$\ell_4 = \text{perpendicular } (S, \ell_3)$;

$Q = \text{meet } (\ell_4, \ell_1)$.

Prove that P and Q are inverses with respect to $\mathcal{C}$.

[10]6. Let Σ be a circle with centre O. Let A be a point interior to Σ, $A \neq 0$. Give a construction for a circle Γ that is orthogonal to Σ and so that the inverse of A with respect to Γ is O.

[10]7. Let Σ be a circle with centre O and P be any point inside Σ, $P \neq 0$, $P \notin \Sigma$. Let L be the perpendicular bisector of the two points P and P^{Σ}. Let X be any point on L. Prove that the circle with center X and radius point P is orthogonal to Σ.

8. Let $\mathcal{S}$ be the sphere $\{(u, v, w) : u^2 + v^2 + (w-1)^2 = 1\}$. Let $N = (0, 0, 2)$ on $\mathcal{S}$. Let π be the plane $\{(x, y, 0) : x, y \in \mathbb{R}\}$, tangent to $\mathcal{S}$ at $(0, 0, 0)$. Consider stereographic projection from the point N.

[5] (a) Given the point (u, v, w) on $\mathcal{S}$, find $(x, y, 0)$ on π that is the projection of (u, v, w) from N.

[5] (b) Given $(x, y, 0)$ on π, find the coordinates of (u, v, w) on $\mathcal{S}$ that is the projection of $(x, y, 0)$ from the point N.

[10]9. For $i = 1, 2$, let $H_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$ represent two circles and $\Delta_i = \det(H_i)$. Let $2\Delta_{12} = A_1D_2 + A_2D_1 - B_1C_2 - B_2C_1$. Suppose that ω is the angle between the two circles. Suppose also that $\Delta_1\Delta_2 - (\Delta_{12})^2 \geq 0$. Prove that $-1 \leq \cos\omega \leq 1$.

[3]10. (a) A circle is given by the Hermitian matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Write the formula for inversion with respect to this circle.

[3] (b) For the particular circle

$$(x-5)^2 + (x-3)^2 = 16$$

find the Hermitian matrix that represents it.

[4] (c) Using (a) and (b), or otherwise, find the formula that gives the inverse of the complex point z with respect to the particular circle given in (b).