PMath 360 Sample Final 2000

1. **Homogeneous Coordinates**

   Let P, Q, R, and S be points given by

   $$P : p = (p_1, p_2, p_3), \quad Q : q = (q_1, q_2, q_3), \quad R : r = (r_1, r_2, r_3) \text{ and } S : s = (s_1, s_2, s_3).$$

   For each of the following geometric statements about points, give an equivalent algebraic statement about their coordinate vectors.

   [2] (a) The points P and Q are equal.

   [3] (b) The points P, Q, and R are distinct and collinear.

   [5] (c) The points P, Q, R, and S form a frame.

2. **Collineation**

   Find the matrix of the collineation that maps the standard frame to the frame

   $$F_1 : (2, 0, 3), \quad F_2 : (0, 1, 5), \quad F_3 : (2, 1, 0) \quad F_4 : (2, 8, 3).$$

3. **All tangents to a conic**

   Let Γ be the conic whose equation is

   $$3x_1^2 + 2x_1x_2 + x_2^2 - 2x_3^2 = 0$$

   [2] (a) Find the matrix that represents Γ.

   [2] (b) If $X : x = (x_1, x_2, x_3)$ is a point on Γ, and $M : m = [m_1, m_2, m_3]$ is the line tangent to Γ at X, then the vectors x and m are related by what equation?

   [6] (c) Find the equation involving the coordinates of m that is necessary and sufficient for the line M to be tangent to Γ.

4. **Matrix of a conic**

   Let Γ be the non-degenerate conic that

   - contains $(1, 0, 0)$,
   - is tangent to $[2, 0, 5]$ at $(0, 1, 0)$, and
   - is tangent to $[2, 3, 0]$ at $(0, 0, 1)$.

   [8] (a) Find the matrix of Γ.

[2] (b) Find the coordinates of the line tangent to Γ at $(1, 0, 0)$.

5. **Polarity**

A polarity of the plane is given by $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & -3 \\ 4 & -3 & 0 \end{bmatrix}$.

[1] (a) Write the Cartesian equation of the conic associated with this polarity.

[2] (b) Find the polar line L of the point $(3, 4, -10)$.

[4] (c) Find the coordinates of S and T, the points that are the intersection of L with Γ.

[3] (d) Find the point of intersection of the tangents to Γ at S and T respectively.

6. **Inversion in the plane**

A point P and a circle Σ are given. P is not at the centre of Σ. Give a construction for the inverse of P with respect to Σ. Prove that your construction is correct.

7. **Cirlces in the Complex Plane**

Two circles C_1 and C_2 are given.

C_1 has centre $(3, 4)$ and radius 5

C_2 has centre $(2, 1)$ and radius 1.

[3] (a) Find the Hermitian matrices H_1 and H_2 that represent C_1 and C_2, respectively.

[4] (b) Find the cosine of the angle determined by H_1 and H_2.

[3] (c) Find an Hermitian matrix that represents the line in the pencil determined by H_1 and H_2.

8. **Stereographic Projection**

Let $\mathcal{S}$ be the sphere $\{(u, v, w) : u^2 + v^2 + w^2 = 1\}$.

Let $S : (0, 0, -1)$. Consider the stereographic projection from the plane $(x, y, 0)$ to the sphere $\mathcal{S}$, using S as the centre of projection.

Let C be the circle in the xy-plane given by $(x - 4)^2 + (y + 3)^2 = 6^2$, and let D be the stereographic projection of C on $\mathcal{S}$.

Find the equation of the plane in 3 space that contains D.

9. **Orthogonal Circles**

[4] (a) Suppose C_1 and C_2 are any 2 circles with centres O_1 and O_2, respectively. Let X be any point not on either circle and not on line (O_1, O_2). Give a construction for a circle $K(X)$ that passes through X and is orthogonal to both C_1 and C_2.

[6] (b) Suppose further that C_1 and C_2 are disjoint. It is a fact that, for any X as described above, $K(X)$ meets line (O_1, O_2) in two points which are independent of X. Let Z be one of these intersection points and let Σ be any circle whose centre is Z. Let D_1 and D_2 be the inverses with respect to Σ of C_1 and C_2, respectively. Prove that D_1 and D_2 have a common centre.