Instructions:

1. There are 8 questions.
2. Show your work in the space provided.
3. If you need more space, continue your work on the back of the previous page or on the blank page at the end, but indicate in the space provided where your work is continued.
4. No calculators or other aids may be used.

Question 1. Projective plane: Desargues' Theorem

State and prove the theorem of Desargues.

Question 2. Projective plane; collineation.

Four points of a frame are given by F_1 : (1, 1, 2), F_2 : (5, 1, 3), F_3 : (1, 2, 4) and F_4 : (0, 6, 1). There is exactly one collineation that maps the standard frame to the frame F_1, F_2, F_3, F_4.

Let P : (1, 2, 3).

(a) Find a matrix that represents this collineation.

(b) Find the image of the point P under this collineation.

Question 3. Projective plane, polarity.

In the Cartesian plane the point P : (-0.4, -0.3) and the conic Γ with equation $x^2 - 6x + 2y^2 + 8y = 0$ are given. Answer these question using homogeneous coordinates and equations, as appropriate.

(a) Write coordinates for P and write a symmetric matrix A for Γ.

(b) Find the line L so that P is the pole of L with respect to Γ.

(c) Find S and T, the two points of intersection of L with Γ.

(d) Find the point where the polars of S and T with respect to Γ meet.

Question 4. Pencil of Circles.

Let F be the pencil of circles containing the circles given by

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & -25 \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} -1 & 3+4i \\ 3-4i & -21 \end{bmatrix}.$$

(a) Write the centre and the radius of each of the two circles.

(b) Write the matrix that represents the line in F.

(c) Find a matrix that represents a circle in F containing the point 1+ i .

(d) Find the cosine of the angle between the two circles given by H_1 and H_2 .

Question 5. Circle constructions.

Two non-intersecting circles C_1 and C_2 are given. Let P be a point in general position, that is, P not on C_1 and P not on C_2 and P not on the line of centres for C_1 and C_2.

(a) State the steps of construction using a straight-edge and compass for the circle through P that is orthogonal to C_1 and C_2.

(b) State the steps of construction for a circle Σ so that C_1 and C_2 are inverted into concentric circles when inverted with respect to Σ.

(c) Using a straight-edge and compass show the figure produced by the steps of your construction on the figure below.

Question 6. Stereographic projection.

Let S be the sphere with center (0,0,0) and radius 1.
Let π be the xy-plane. Consider the stereographic projection (as defined in lecture and in the notes) which maps the point (u,v,w) on the sphere S to the point (u,v,0)/(1+w) in the plane π by projecting through the point S:(0,0,-1).

(a) A circle in π is given by the equation $Az\bar{z} + Bz + C\bar{z} + D = 0$. Use the stereographic projection defined above to derive the equation of the plane in 3-space that meets S in the image of this circle.

(b) Find the equation of the plane that contains the image of the circle in π whose equation is $x^2 + y^2 + 4x - 6y + 12 = 0$.

Question 7. Stereographic projection (again).

Consider another sphere with centre (0, 0, g) and radius g.
Let N be the point on this sphere given by N:(0, 0, 2g) .
Let π be the xy-plane. (It is tangent to the sphere at the origin).

(a) Write the equation of S.

(b) Find the equation(s) for stereographic projection from the point (x,y,0) in the plane to the point (u, v, w) on the sphere.

Question 8. Mutually tangent circles.

Three circles C_1, C_2, and C_3 are given to be mutually tangent, as in the figure below. State steps necessary to find two circles, D_1 and D_2 so that

(D_1 is tangent to C_1, C_2 and C_3) and
(D_2 is tangent to C_1, C_2 and C_3).