Instructions:

➣ Show your work in the space provided. If you need more space, use the back of the previous page or the blank page at the end of the test, and indicate (in the space provided) where your work is continued.

➣ Highest marks are given for well reasoned and clearly presented arguments.

➣ Each question has equal weight.

➣ This test has 8 questions

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1. **Geometry and Algebra**

In this question, we are looking for the algebraic conditions that are equivalent to various geometric statements. The notation X:**x** means that the point X is represented by the vector **x**. Several points P:**p**, Q:**q,** R:**r**, and S:**s** in the real projective plane are given. In each part below, there is a geometric statement about the points. You are to give the equivalent algebraic statement about the cooresponding vectors.

1.1 The points P and Q are equal.

1.2 The points P, Q, and R are distinct and collinear.

1.3 The points P, Q, R, and S form a frame.

2. **Prove**

State and prove the Theorem of Desargues.

3. **Projective Collineation**

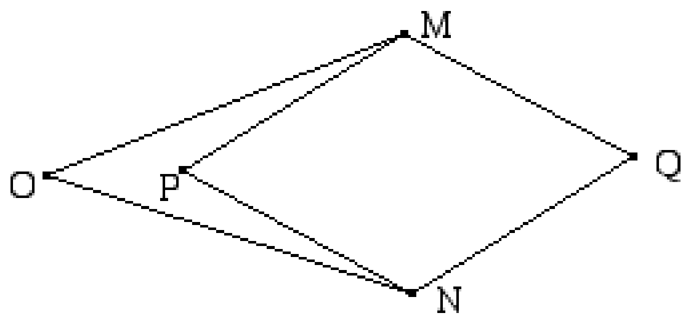
In the real projective plane, the points A:(0,0,1), B:(11,0,10), C:(1,1,1), and D:(0,12,10) are given. Let $E=AB\cap CD$ and $F=BC\cap DA$.

3.1 Find the matrix of the collineation that maps the point at infinity on the x-axis to E and the point at infinity on the y-axis to F, A to itself, and C to itself. Use the usual embedding of the Cartesian plane into the real projective plane, by which (x, y) is mapped to (x, y, 1).

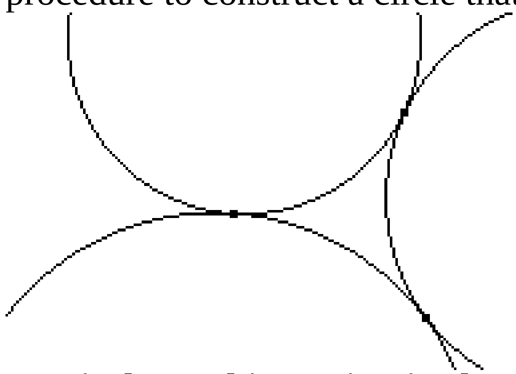
3.2 Explain how (or why) the existence of the above collineation verifies that the four points A, C, E, and F forms a frame.

4. **Inversion linkage**

A linkage is given as in the diagram below. The point O is fixed, and all segments shown have fixed length. The lengths of OM and ON are equal to the number b, and the lengths of the other four segments are equal to the number a, and $a < b$. The point P can be moved about (within certain limits), and when it moves, the other three points M, N, and Q also move. Prove that the points P and Q are inverses of each other with respect to some circle with centre at O.



5. **Application of Inversion**
Three circles are arranged so that each is tangent to the other two. Give a procedure to construct a circle that is tangent to all three.



6. **Circles and inversion in the complex plane**

6.1 Find a complex function $f(z)$ for $z = x + iy$ that represents inversion with respect to the circle given by $(x-2)^2 + (y-3)^2 = 5$.

6.2 Suppose that an Hermitian matrix $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is given to represent a real circle in the complex plane. Write the function $f(z)$ in terms of A, B, C, and D that represents inversion with respect to that circle.

6.3 Supply some evidence that your answer is correct by demonstrating that $f(f(z)) = z$.

7. **Stereographic projection**

Consider the sphere S with centre (0,0,0) and radius 1 and the stereographic projection from S to the complex plane, as developed in class, given by this correspondence $(u,v,w) \leftrightarrow (u+iv)/(1-w)$.

7.1 **From the plane to the sphere**

A circle in the complex plane is given with centre 3+4i and radius 5. Find the plane that contains the circle on the sphere that corresponds to this circle.

7.2 **From the sphere to the plane**

A certain circle on the sphere lies in the plane $3u+12v+4w=1$. Find the circle in the complex plane that corresponds to this.

8. **Circles**

A circle Σ and two points P and Q interior to Σ are given.

8.1 Give the shortest straight-edge and compass construction you know for the inverse of P with respect to Σ.

8.2 Give a straight-edge and compass construction for the circle orthogonal to Σ that contains P and Q.