Question 1

Let A and B be two distinct points interior to a circle Σ and distinct from the centre of Σ. Give a step by step construction for a circle orthogonal to Σ which contains the points A and B. How many such circles are there? (Explain.) What can be said if one of the points, say A, lies at the centre of Σ ?

In questions 2 and 3, let $\mathcal{S} = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}$. Let S:(0,0,-1). Let σ be the stereographic projection of the plane z = 0 to the sphere $\mathcal{S}$ from the point S.

Question 2

Let $\mathcal{D}$ be the circle in the plane z = 0 whose equation is

$$(x - 10)^2 + (y + 13)^2 = 16.$$

(a) Find the equation of the plane δ that contains $\sigma(\mathcal{D})$.

(b) Find D, the pole of δ with respect to the sphere $\mathcal{S}$.

(c) Let Q :(3, 7, 0) and find Q ', the inverse of Q with respect to $\mathcal{D}$ in the plane z=0.

(d) Show that σ(Q), σ(Q '), and D are collinear.

Question 3:

In the plane z = 0, let $\mathcal{D}_1$ be the circle whose equation is

$$(x - 10)^2 + (y + 13)^2 = 16 .$$

and let $\mathcal{D}_2$ be the circle whose equation is

$$(x + 4)^2 + (y - 5)^2 = 9.$$

(a) Sketch $\mathcal{D}_1$ and $\mathcal{D}_2$ to verify that they are non-intersecting circles.

(b) Let $\mathcal{D}_3$ be the circle whose equation is

$$11 [(x - 10)^2 + (y + 13)^2 - 16] - 9 [(x + 4)^2 + (y - 5)^2 - 9] = 0 .$$

Prove that $\mathcal{D}_3$ does not have any points in common with either of the circles $\mathcal{D}_1$ or $\mathcal{D}_2$.

(c) Let δ_i be the plane that contains the circle $\sigma(\mathcal{D}_i)$, for i = 1, 2, 3. Show that the line δ_1 intersect δ_2 does not meet the sphere $\mathcal{S}$.

(d) Show that the plane δ_3 contains the line δ_1 intersect δ_2.

Question 4

Verify that (1,2,1), (1,3,1), (-3,3,3), (2,0,0) is a frame. Find the change of coordinate map that takes the standard frame (1,0,0), (0,1,0), (0,0,1), (1,1,1) to the above frame.

Question 5

An archaeologist has just returned from a dig with a photograph of a square plot showing the position of several important artifacts. Unfortunately, the camera was aimed in such a way that the square plot appears in the picture to be an irregular quadrilateral. Measurements can be taken on the photo, and one of the edges may be thought of as an x axis, but none of the other sides seem to fall nicely into place.

The archaeologist has asked if it is possible to find a function that maps the interior points of the quadrilateral in the photograph onto the interior points of the unit square. See Figure 1. [Some data was in the Figure. It is lost.]

Transform this question to one about homogeneous coordinates, and find the mapping.

Question (6)

Let P, Q, R, and S be points given by P:(p_1, p_2, p_3), Q:(q_1, q_2, q_3), R:(r_1, r_2, r_3) and S:(s_1, s_2, s_3) .

For each of the following statements, state a condition on the coordinates that is both necessary and sufficient for the statement to be true. Explain, or prove, in each case.

(a) The points P and Q are equal.
(b) The points P, Q, and R are distinct and collinear.
(c) The points P, Q, R, and S form a frame.

Question (7)

There are 24 permutations of four symbols a, b, c and d. For most values of a, b, c, and d, exactly 4 of these permutations give the same cross ratio as (a, b; c, d). However, if it happens that (a, b; c, d) = -1, there are another four permutations which give the same value, giving a total of eight permutations with this same cross ratio.

Let A, B, C and D be four points on a line arranged in such a way that A is the harmonic conjugate of B with respect to C and D. Then (A, B; C, D) = -1. Find sequences of perspectivities which permute the four points. Since the cross ratio is invariant under perspectivities, there will be at most eight such permutations. There may be more than one sequence of perspectivities for any one permutation, but in your solution, present at most one sequence for each permutation. Figure 2 might help. [Sadly, Figure 2 iis lost, but it might have been a lot like the figure in the notes on page 112.]

Question (8)

Express the mapping

$$x \;\text{--->}\; (ax+b) \;/\; (cx+d)$$

as a composition of mappings of these three types:

x ---> mx,
x ---> $x+k$,
x ---> $1/x$.

Are there any choices of a, b, c and d such that your composition is either singular or not defined? If so, can you find another sequence of maps of the above three types that expresses the map in this case?

Question (9)

Let A:(0,1,1), B:1,1,0), C:(2,2,2) and V:(1,0,1). Find the equation of the non-degenerate conic which contains the three points A, B, and C, and has the lines VA and VB as tangent lines.

Question (10)

Let G be the conic whose Cartesian equation is

$$x^2 + 2xy + 5y^2 - 4 = 0.$$

Express this equation with homogeneous coordinates in matrix form. Let L be a line whose coordinates are [a, b, c]. Find an equation for a, b, and c which is a necessary and sufficient condition for L to be tangent to G.