North Pole Stereographic Projection
A note for PMath 360

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1 Stereographic Projection

1.1 Introduction

Definition 1. Stereographic projection
A **stereographic projection** is a mapping of objects in three dimensions from one surface to another. Usually it is from a plane to a sphere or from a sphere to a plane, and various authors differ on which way this should be. Here, through the use of particular example, we will show that there is a way to think about these two mappings (that *are* inverses of each other) as the restriction of yet another mapping that can be restricted to either one or the other of the two stereographic projections.

The following objects will be used in the three examples of mappings in this section.

- Let $\mathbb{C}$ be the set of complex numbers, with typical elements of the form $z = x + iy$.

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- Let γ be the u,v-plane in $\mathbb{R}^3$. Its equation is $w = 0$, and its points have the form $(u,v,0)$.
- Identify each complex number $z = x + iy$ with the point (x, y, 0) in γ.
- Let $\mathscr{S}$ be the sphere with center (0,0,0) and radius 1. Then the set of points on the sphere $\mathscr{S}$

$$\mathscr{S} = \{(u,v,w) : u^2 + v^2 + w^2 = 1\}.$$

- Let $\mathscr{S}$ be the sphere with center (0,0,0) and radius 1. The points on $\mathscr{S}$ satisfies the equation

$$u^2 + v^2 + w^2 = 1.$$

- Let N:(0,0,1) be the **center of projection** on $\mathscr{S}$.
- Let $\mathscr{S}\backslash N$ denote the points of the sphere $\mathscr{S}$ that are distinct from N.

1.2 σ_1: stereographic projection from $\mathscr{S}\backslash N$ to $\mathbb{C}$

Let us define a mapping $\sigma_1 : \mathscr{S}\backslash N \mapsto \mathbb{C}$.

Let P:(u,v,w) be any point on $\mathscr{S}$ distinct from N. The image $\sigma_1(P)$ is defined to be the point $z = x + iy$ in the complex plane that is identified with the point Q:(x, y, 0) on the plane γ such that Q, P, and N are collinear. Since Q, P and N are collinear, there is a s real number t so that

$$(x, y, 0) = t(u,v,w) + (1-t)(0,0,1).$$

Looking at the third coordinate, we see 0 = t w + (1-t). Solving for t we find t = 1/(1-w). Thus

$$P : \left(\frac{u}{1-w}, \frac{v}{1-w}, 0\right) \text{ and } z = \frac{u+iv}{1-w}.$$

$$\boxed{\sigma_1 : (u,v,w) \mapsto \frac{(u+iv)}{(1-w)}}$$

1.3 σ_2: Stereographic projection from $\mathbb{C}$ to $\mathscr{S}\backslash$N

Let $z \in \mathbb{C}$. Then z is identified with the point $Z : (x, y, 0) \in \gamma$. We want to find the point $P : (u, v, w)$ so that:

- P is on the sphere $\mathscr{S}$,
- P is on the line ZN,
- P is not $\mathscr{S}$.

As a consequence, we need to find a number k so that

$$\mathscr{S}\, u^2 + v^2 + w^2 = 1, \tag{1}$$
$$P : (u, v, w) = j(x, y, 0) + (1 - j)(0, 0, 1) \tag{2}$$
$$j \neq 0. \tag{3}$$

From the first two conditions, we get the equation

$$(jx)^2 + (jy)^2 + (1 - j)^2 = 1$$

After subtracting 1 from both sides and factoring j, we have

$$j(jx^2 + jy^2 - 2 + j) = 0.$$

Since $j \neq 0$, we solve $jx^2 + jy^2 - 2 + j = 0$ for j and get

$$j = \frac{2}{x^2 + y^2 + 1}$$
$$1 - j = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

Then, using Equation (2) we get

$$P : \big(x, \;\; y, \;\; x^2 + y^2 - 1\big) / (x^2 + y^2 + 1) \tag{4}$$

In summary,

$$\sigma_2 : x + iy \mapsto \left(\frac{x}{x^2 + y^2 + 1}, \frac{y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right)$$

$$\boxed{\sigma_2 : x + iy \mapsto \frac{1}{x^2 + y^2 + 1} \big(x, y, x^2 + y^2 - 1\big)}$$

1.4 σ_1 and σ_2 are inverses of each other

You have, I hope, recognized that σ_1 and σ_2 are inverses of each other. Perhaps this is not evident from the final *equations* of these three dimensional functions, but geometrically their definitions make it obvious. Consider these points:

- In both cases we have a projection with center N.
- In both cases, $Z \in \mathbb{R}^3$ corresponds to $z \in \mathbb{C}$.
- In both cases, we have three collinear points: N, P and Z.
- In the case of σ_1 we are given $z \in \mathbb{C}$ and we find $P \in \mathscr{S}$.
- In the case of σ_2, we are given $P \in \mathscr{S}$ and we find $z \in \mathbb{C}$.

Thus, $\sigma_2 = \sigma_1^{-1}$.

1.5 Inversion in $\mathbb{R}^3$

Suppose that Σ is a sphere in three space and it has center C and radius r. The mapping that takes the point P to the point P', where P' lies on ray(C,P) and $|CP|\,|CP'| = r^2$ is called inversion with respect to the sphere Σ. Recall that in the plane, the inverse of the center of a circle of inversion is not defined. Likewise, in $\mathbb{R}^3$ the inverse of the center of the sphere is not defined.

Many of the properties that inversion with respect to a circle in the plane have analogs for inversion with respect to a sphere in three space.

Here are some, which we state without proof:

1. Planes not on the center C are mapped to spheres that pass through C.
2. Spheres on C are mapped to planes not on C.
3. Planes on C are inverted into themselves.
4. Spheres not on C are inverted to spheres not on C.
5. Spheres orthogonal to Σ are inverted to themselves.
6. Incidence is preserved.

2 A particular inversion in $\mathbb{R}^3$

Let Σ be the sphere with center at N and radius point any point on the equator $\gamma \cap \mathscr{S}$. The point (1,0,0) will do. This makes the radius $\sqrt{2}$ for Σ.

Let σ denote inversion with respect to the sphere Σ.

If we restrict the domain of σ to points $z \in \mathbb{C}$, or points Z in γ, then inversion σ has exactly the same effect as the mapping σ_1.

Likewise if we restrict the domain of σ to points P on the sphere, the map σ has exactly the same effect as σ_2.

Because of this we see that not only are σ_1 and σ_2 inverses of each other, but there is one function, namely σ that agrees with both σ_1 and σ_2. and with that in mind, they are both parts of one function, inversion with respect to the sphere Ω.

Moreover, we understand that stereographic projection is not limited to a one way map from a plane to a sphere or from a sphere to a plane, but it is part of a bigger idea, that of a two way map that is inversion of three space with respect to a sphere Ω. Because the sphere $\mathscr{S}$ passes through the center of Ω the image of $\mathscr{S}$ is a plane, and in this case it is the plane γ. Similarly, because γ is a plane, it is inverted to a sphere through the center of Ω.

In fact, γ and $\mathscr{S}$ are inverses of each other with respect to the three dimensional inversion with respect to Ω.

3 A particular circle in $\mathbb{C}$

Consider any point $P : (u, v, w) \in \mathscr{S}$. Let H be an Hermitian matrix that represents a circle in $\mathbb{C}$. Suppose

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In order for the point $\sigma_1(P)$ to be on the circle given by H, the coordinates of $\sigma_1(P)$ must satisfy the equation represented by H. Let

$$\mathrm{z} = \sigma_1(P) = \frac{(u + iv)}{(1 - w)}.$$

The following are equivalent:

The point P on the sphere $\mathscr{S}$ corresponds to a point z on the circle given by H

$$\begin{gathered}
(z, 1) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \overline{z} \\ 1 \end{pmatrix} = 0 \\
Az\overline{z} + Bz + C\overline{z} + D = 0 \\
A\frac{(u+iv)}{(1-w)}\frac{(u-iv)}{(1-w)} + B\frac{(u+iv)}{(1-w)} + C\frac{(u-iv)}{(1-w)} + D = 0 \\
A\frac{(u^2+v^2)}{(1-w)(1-w)} + B\frac{(u+iv)}{(1-w)} + C\frac{(u-iv)}{(1-w)} + D = 0 \\
A\frac{(1-w^2)}{(1-w)(1-w)} + B\frac{(u+iv)}{(1-w)} + C\frac{(u-iv)}{(1-w)} + D = 0 \\
A\frac{(1-w)(1+w)}{(1-w)(1-w)} + B\frac{(u+iv)}{(1-w)} + C\frac{(u-iv)}{(1-w)} + D = 0 \\
A(1+w) + B(u+iv) + C(u-iv) + D(1-w) = 0 \\
(B+C)u + i(B-C)v + (A-D)w + (A+D) = 0.
\end{gathered}$$

This final equation, we recognize as the equation of plane in $\mathbb{R}^3$ because all four of the coefficients are real: B+C is real because B and C are complex conjugates of each other. The same is true for i(B-C). Finally, both A-D and A+D are real because A and D are real.

If we let

$$\begin{aligned} a &= B + C, & c &= A - D, \\ b &= i(B - C), & d &= A + D, \end{aligned}$$

then we see that the stereographic image of the circle given by the Hermitian matrix H is the intersection of the sphere $\mathscr{S}$ and the plane whose equation is $au + bv + cw + d = 0$.

The End

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