11 Plane $\Leftrightarrow$ Sphere

In this section, let $\mathcal{S}$ be the sphere in $\mathbb{R}^3$ with centre (0,0,0) and radius 1. Let $N : (0,0,1) \in \mathcal{S}$ be the point of projection on $\mathcal{S}$ so that a point $z = x + iy$ in the complex plane corresponds to a point $(u, v, w) \in \mathcal{S}$ if and only if the three points $N : (0,0,1), (u,v,w)$ and $(x,y,0)$ are collinear. (This is the stereographic projection presented in lecture.)

11.1 An Hermitian matrix H_1

Consider the Hermitian matrix

$$H_1 = \begin{pmatrix} 1 & -3-4i \\ -3+4i & -11 \end{pmatrix}.$$

11.1.1 (*) Centre and radius

In the complex plane, find the centre and radius of the circle given by the equation

$$(z,1)H_1 \begin{pmatrix} \overline{z} \\ 1 \end{pmatrix} = 0$$

11.1.2 (*) The plane δ_1

Find the equation of the plane δ_1 in $\mathbb{R}^3$ such that the intersection $\delta_1 \cap \mathcal{S}$ is the projection of H_1 on the sphere $\mathcal{S}$.

11.2 A second Hermitian matrix H_2

Consider a second matrix H_2 given by

$$H_2 = \begin{pmatrix} 1 & 5-4i \\ 5+4i & -99 \end{pmatrix}$$

and consider the one parameter family of circles given by $\lambda_1 H_1 + \lambda_2 H_2$.

11.2.1 (*) A line

Find values of λ_1 and λ_2 that give the straight line in this family. Write the matrix and the Cartesian equation of this line.

11.2.2 (*) Δ_1, Δ_2, and $\Delta_{1,2}$

Find the values Δ_1, Δ_2, and $\Delta_{1,2}$ associated with the quadratic form

$$\det\left(\lambda_1 H_1 + \lambda_2 H_2\right) = \Delta_1 \lambda_1^2 + 2\Delta_{1,2}\lambda_1\lambda_2 + \Delta_2\lambda_2^2$$

11.2.3 (*) $\cos(\omega)$

Find the value of the cosine of the angle determined by the directed circles H_1 and H_2 as a function of Δ_1, Δ_2 and $\Delta_{1,2}$.

11.2.4 (*) $\Delta_1\Delta_2 - (\Delta_{1,2})^2$

Find the value of the discriminant $\Delta_1\Delta_2 - (\Delta_{1,2})^2$

Footnote

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(*) Items marked with an asterisk should be submitted for marking.

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