# 10 Train, Necklace and Porism

## 10.1 Train of circles

### 10.1.1 Foundation line

Let $L$ be any line. A horizontal line will do. Hide the point(s) used to define $L$. Let $P_1$ and $P_2$ be any two, new, distinct points on $L$. Let and $L_1$ and $L_2$ be lines perpendicular to $L$ at $P_1$ and $P_2$, respectively.

Confirm for yourself that $P_1$ and $P_2$ may be used as "handles" on $L$ (that does not move) to move $L_1$ and $L_2$ (which stay parallel to each other) .

### 10.1.2 A train of circles between $L_1$ and $L_2$

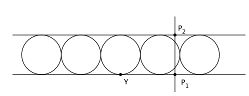
Let $Y$ be a point on $L_1$ , ($Y$ distinct from $P_1$). Find the circle $D_0$ that passes through Y and is tangent to $L_1$ (at $Y$) and to $L_2$.

Find two distinct circles $D_1$ and $D_{-1}$ both tangent to lines $L_1$ and $L_2$ and tangent to the circle $D_0$.

Find two more new circles $D_2$ and $D_{-2}$ both tangent to $L_1$ and $L_2$. We want $D_2$ tangent to $D_1$ but not equal to $D_0$ and $D_{-2}$ is a distinctly new circle tangent to $D_{-1}$ but not to $D_0$. You should have 5 circles in a row.

Confirm for yourself that as you move $Y$, all five circles move so that they are all tangent to $L_1$ and $L_2$ and each continues to be each tangent to the next.

( Comment: Of course one could construct many more D circles, but *knowing* that this can be done is enough. Don't get carried away.)




### 10.1.3 Two tangent circles

Let $P$, $Q_1$ and $Q_2$ be three new points on $L$. Find circles $C_1$ and $C_2$ so that, for $i = 1, 2$, $C_i$ has the segment $PQ_i$ as a diameter. Confirm for yourself that $C_1$ and $C_2$ are tangent to each other at $P$, no matter how $P$, $Q_1$ and $Q_2$ are moved on $L$.

### 10.1.4 Circle of inversion

Find any circle $\Sigma$ with center at $P$. Construct $Q_1' = \text{inv}(Q_1,\Sigma)$ and $Q_2' = \text{inv}(Q_2,\Sigma)$. redefine $P_1$ to be $Q_1'$. See the footnote.[1] Similarly, redefine $P_2$ to be ${Q'}_2$ .

Confirm for yourself that $Q_1$ is in control of not only $C_1$ but also $P_1$ and $L_1$. Confirm for yourself that $L_1$ and $C_1$ are inverses of each other with respect to $\Sigma$. Do similarly for $Q_2, C_2$ and $L_2$.

### 10.1.5 Control point $X$

Let $X$ be any point on $C_1$. Let $X' = \text{inv}(X, \Sigma)$. Confirm for yourself that as you move $X$ on $C_1$, $X'$ moves on $L_1$ . This is positive feed back that confirms that $L_1$ is the inverse of $C_1$.

Redefine $Y$ to be $X'$. Confirm for yourself that now $X$ on $C_1$ controls the train of circles that lie on the track $L_1$ and $L_2$.

### 10.1.6 Invert the train

Invert the five circles $D_{-2}$, $D_{-1}$, $D_0$, $D_1$, $D_2$ with respect to $\Sigma$. Animate $X$ on $C_1$. (Perhaps you still have a macro to invert one circle with respect to another.) Confirm for yourself that you have a train of circles between $C_1$ and $C_2$, tangent to $C_1$ and $C_2$, and controlled by $X$, which is on the middle

[1] In the GeoGebra toolbar at the top of the screen, select the pop down menu called **Edit**. In that menu select the item "Object Properties." This opens a new, smaller window.

- In the Object Properties window, look for the word **Points**. Click on the triangle next to the word point to see names of all the points in your figure. Select the point $Q_1^{'}$ and note its definition in the line containing the words `Definition: Reflect(Q`$_1$`,`$\Sigma$`).` Capture this text, if you can.
- Edit the point $P_1$ and make its definition the same as $Q_1$ was/is. $P_1$ will move to the same position as $Q_1'$ taking $L_1$ with it.
- It should be safe to delete $Q_1'$. A quick and dirty test for correctness: Just move $Q_1$ about, and see if $P_1$ moves the way you think it should.

Just by moving $Q_1$ about, verify that $P_1 = \text{ inv } (Q_1, \Sigma)$.

on of the set of 5. It appears that the control between $X$ and these five unnamed circle is direct.

#### 10.1.7 Explain (*)

Tidy your figure by hiding intermediate construction objects. Submit your figure, showing $\Sigma$, $C_1$, $C_2$, $L_1$ , $L_2$, the train of 5 circles between the two tracks, and their inverses that lie between $C_1$ and $C_2$.

**10.1.7.1 (*)** Submit your source code.

**10.1.7.2 (*)** In your own words, explain what is going on here.

In your explanation, address these questions: (i) What is the statement of the problem? (ii) How was the problem solved?

## 10.2 Necklace of circles, a Porism

### 10.2.1 Warm-up

Let $C_u$ and $C_v$ be distinct circles having a common center $o$.

Create $u$, a point on $C_u$.

Construct a circle $D_1$ that is between $C_u$ at $u$ and also tangent to $C_v$, and $D_1$ is tangent to $C_u$ at $u$, and it is also tangent to $C_v$. Let $d_1 = \text{center}(D_1)$.

Construct the point $t$ on $D_1$ so that line $T = \text{line}(o,t)$ is tangent to $D_1$. Test your construction by expanding the outside circle and shrinking the inside circle so that the radius is nearly zero. If your line $T$ still looks like it is tangent to $D_1$ you will have avoided one of the common mistakes and it is probably correct.

Let $d_2$ be the reflection of $d_1$ in the line $T$ and $u_2$ be the reflection of $u$ in T.

Let $D_2 = \text{circle}(d_2, u_2)$. $D_2$ should be tangent to $D_1$ at $t$, to $C_u$ at $u_2$ and to $C_v$.

Create a macro depending on $C_u$, $C_v$ and $u$ that produces final objects $D_2$ and $u_2$.

Use your macro to make several circles, D3, D4, and so on, each tangent to the next, and all tangent to $C_u$ and all tangent to $C_v$.

Your construction should have these properties:

1. Changing the size of $C_u$ or $C_v$ modifies the sizes of $D_1$, $D_2$, ... and so on, but the size of $D_1$ will be the same as the size of each of the circles $D_2$, $D_3$, ... and so on. The $D$ circles are said to form a train of circles between $C_1$ and $C_2$.
2. Moving $u$ on $C_u$ moves the train of circles between $C_u$ and $C_v$.
3. If the last circle is tangent to the first circle, then moving the point $u$ on $C_u$ preserves that particular tangency. If the first and last circles are not tangent, then that condition is also preserved as $u$ moves on $C_u$.

End of the warmup.

### 10.2.2 Construction

Start again with a fresh figure.

Let $C_1$ and $C_2$ be any two non-intersecting circles with $C_2$ completely interior to $C_1$.

Find two circles $\Gamma_1$ and $\Gamma_2$, both orthogonal to $C_1$ and to $C_2$. If you like, one of these could be the line of centers of $C_1$ and $C_1$.

At this point, what you learned in exercise 8.3 and 8.6 come into play. Let A and B be the points of intersection of $\Gamma_1$ and $\Gamma_2$ Using either A or B as center, construct a circle $\Sigma$ that inverts $C_1$ and $C_2$ to two concentric circles. Call the concentric circles $C_1'$ and $C_2'$.

Let x be a point on $C_1$. Let $x' = \text{inv}\ (x, \Sigma)$, and note that it is on $C_1'$. Now make use of what you learned in the warmup to produce a set of circles, $D_1'$, $D_2'$, $D_3'$, ... $D_7'$, starting with $D_1'$ on $x'$, so that so that each is tangent to the next, and so that all are tangent to $C_1'$ and to $C_2'$. Also, these should be under the control of $x$, since they were constructed so that $D'_1$ is tangent to $C'_1$ at $x'$.

Depending on how many circles are in the chain and the sizes of $C_1$ and $C_2$, the chain might or might not loop back to itself, making a complete and connected loop. If they begin to wrap around on top of each other, adjust the center or the radius of $C_1$ or $C_2$ or both so that the D' circles become smaller so that there is no overlap, at least during this construction phase.

Find $D_i = \text{inv}(D'_i,\Sigma)$, the inverses of $D'_i$ with respect to $\Sigma$, for $i = 1, 2, ...$ 7.

(Checkpoint: If you got it right, each of the seven new circles is tangent to the next, and all are all tangent to $C_1$ and $C_2$.)

Now adjust the size of $C_1$ and or $C_2$ or both, if necessary, or the centres of $C_1$ or $C_2$ or both, until circles $D_7$ and $D_1$ appear to be tangent to each other. At the same time, you should notice that $D'_7$ and $D_1$' also appear to be tangent to each other. (There is high probably that we are dealing with some approximation to tangency here, with $D_7$ and $D_1$ not really tangent, they just appear to be. However, assuming you built $D_1$', $D_2$', ... $D_7$' correctly the rest of the points of contact are true points of tangency.)

Check your work by animating x and watch your seven circles move around between $C_1$ and $C_2$, preserving tangency all the time.

### 10.2.3 Print (*)

Hide any construction details that detract from the appearance of your figure. Submit your figure and the source code.

### 10.2.4 Vocabulary

For the purposes of this exercise, let $C_1$ and $C_2$ be non-intersecting circles. and define notions of **train** and **necklace** as follows.

We say that the set of circles $D_1$, $D_2$, ... is a **train of circles** if each circle $D_i$, $i = 1$, 2, ... , is tangent to both $C_1$ and $C_2$, and, for each $j$, $j$=2, ..., each circle $D_j$ is tangent to $D_{j-1}$.

We will say that a train of circles is a **necklace of length n** if the circles (i) $D_1$, for $i$=1, ... $n$, are distinct and (ii) $D_n$ is tangent to $D_1$.

We will say that the train starts at $x$, if $C_1$ and $D_1$ are tangent to to each other at $x$.

## 10.3 A Porism (*)

Using what you have learned in Question 10.2, give a complete explanation of why the following Porism is true.

> **Porism**
> Let $C_1$ and $C_2$ be two non-intersecting circles. Let x and y be any two points on $C_1$.
>
> If there is a necklace of circles of length $n$ between $C_1$ and $C_2$ that starts at x, then there is a necklace of circles of length n between $C_1$ and $C_2$ that starts at y.

Footnote

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(*) Items marked with an asterisk should be submitted for marking.