9 More reflection and inversion

9.1 Poincaré reflection

Let Ω be the circle that bounds a Poincaré model of the hyperbolic plane. Let O be the centre of Ω. Let A and B be two P-points in this hyperbolic plane. Suppose A, B, and O are not collinear.

In the following we ask for two constructions in the Poincaré plane: one for the P-line connecting two points A and B, and one for the P-line that is the perpendicular bisector between two points.

9.1.1 (*) $L_1 =$ `P-line (A, B)`

Construct P-line on A and B.[1] Make a Tool for the P-line (A, B) with respect to Ω. Submit your Construction Protocol.

These 5 items (i) through (v) are not to be submitted for marking. They are intended to allow you to sharpen your mental tools by getting some experience by using a Geogebra tool. Construct several Poincar lines, (i) some parallel (that is, “meeting” at a point on Ω), (ii) some non-intersecting, (iii) some intersecting. (iv) Construct a triangle with a very small angle sum. (v) Construct a triangle with angle sum zero?

[1] Hints for the P-line(A,B):

- It is an arc that lies on the circle Γ that passes through A and B and is orthogonal to Ω.
- Γ also lies on the point $A' = A^{\Omega}$
- The line segment connecting the centres of Ω and Γ meets Γ in a point on the P-line.
- The end points of the arc lie on Ω and are *not* P-points. If you can, show the end points of the arc as open circles.

9.1.2 (*) $L_2 =$ `P-pbis (A, B)`

Find a construction for a second P-line called L_2 that reflects (inverts) A to B and B to A.[2]

When you find your construction, use software tools to verify to your satisfaction that your work is right. Move A about and watch L_1 and L_2 move, always orthogonal to Ω and to each other.

9.2 Inversions in the complex plane

If the Hermitian matrix H and the function f are given by

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad f(z) = \frac{-C\bar{z} - D}{A\bar{z} + B}$$

so that the matrix H represents the circle with equation $(z, 1)H(\bar{z}, 1) = 0$, then the function f gives inversion with respect to the same circle. That is, for any z in the plane (except for the centre of the circle) the pair z and $f(z)$ are inverses with respect to the circle represented by the matrix H.

9.2.1 Σ, H_1, f_1

Let Σ be the circle in the complex plane with centre 0 and radius 10.

9.2.1.1 (*) Find H_1, the Hermitian matrix of Σ. (*)

9.2.1.2 (*) Use the entries in H_1 to write the function f_1 that gives inverses of points with respect to Σ. (*)

[2]Hints: Since the reflection in L_2 maps A to B and B to A, it must reflect the P-line L_1 to itself. Thus the circle Δ that defines L_2 must also have its centre on the (Euclidean) `line(A,B)`. Look for another pair of points that must be mapped to each other to find the centre of Δ. The "points at infinity" on L_1 must also map to each other. When you have the centre, let *Delta* be the circle with this centre and orthogonal to Ω and hence also to Γ.

9.2.1.3 (*) Verify that the **fixed points**[3] of f_1 are exactly the points that satisfy the equation determined by the matrix H_1. That is, show that

$$z = f_1(z) \iff (z, 1)H_1(\bar{z}, 1)^t = 0.$$

9.2.2 (*) Γ, H_2, f_2

Let Γ be circle with centre (-4, 3) and radius 5. Let H_2 be the matrix, representing Γ. Let f_2 be the function for inverse points with respect to Γ. Do the same three tasks for H_2 and f_2 that you did for H_1 and f_1 in 9.2.1.

9.2.3 The product $\Sigma \cdot \Gamma \cdot \Sigma$

9.2.3.1 Recall that Γ goes through the centre of Σ. Convince yourself that the product of the three inversions, first with respect to Σ, then with respect to Γ, and then with respect to Σ again, might be expected to be a reflection with respect to a line.

9.2.4 The composition $f_3 = f_1 \circ f_2 \circ f_1$

9.2.4.1 (*) Simplify the composition $f_3(x) = f_1(f_2(f_1(z)))$.

9.2.4.2 (*) Use f_3 to write the corresponding matrix H_3.

9.2.4.3 (*) Verify that the fixed points of f_3 are the points of a line. Write the equation of that line.

(*) Please submit items flagged with an asterisk for marking.

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[3] The phrase "z is a **fixed point** of f" means that $z = f(z)$. The reason is that we sometimes think of a function as causing or representing motion. If something does not move, it is said to be *fixed*.