8 Pencils of circles
Poincaré model of Hyperbolic Plane

This set of questions is intended to give you experience with the three types of pencils (families) of circles. There are also examples of mutually orthogonal pencils of circles. In 8.2 you will be studying two pencils of circles in which each circle in one pencil is orthogonal to every circle in the other. In questions 8.3 and 8.4, knowledge about circles in the Euclidean plane are applied to the Poincaré model of the real hyperbolic plane. Questions 8.5 and 8.6 focus on families of circles in the Euclidean plane.

8.1 A family of circles

Build two large circles each outside the other. Label them H3 and H4. Using the tool "Invert Object about Circle", construct the following objects:

- the inverse of H4 with respect to H3, and label it H2;
- the inverse of H3 with respect to H2, and label it H1;[1]
- the inverse of H2 with respect to H1, and label it H0;

Then construct:

- the inverse of H3 with respect to H4, and label it H5 ;
- the inverse of H4 with respect to H5, and label it H6 ;
- the inverse of H5 with respect to H6, and label it H7 ;

Don't get carried away, you don't need more than these to learn what there is to be learned here.

Study (play with) the pencil (family) of circles H0, H1, H2 ... H7. as you move circles H3 and H4 to various positions with respect to each other. Select whichever circle is smaller, H3 or H4, and move it inside the other. Watch the family as they change from being disjoint circles to tangent circles and then to overlapping circles. When the circles are disjoint, they are said to be

[1] If your circles get too small, increase the size of H3 and H4 and move them so the are almost touching.

members of **hyperbolic pencil**, when the circles form a tangent family, a **parabolic pencil**, and they form an overlapping family, an **elliptic pencil**. Notice that in the first category, no two circles meet; in the second category, all circles are tangent to each other the same point, an in the third category, there are two distinct points that are on all the circles in the pencil.

Name	Type	Intersection count
Hyperbolic	disjoint	0
Parabolic	tangent	1
Elliptic	intersecting	2

8.2 Two Families of circles

For i=1,2, let C_i be a circle with centre o_i and radius point r_i. For simplicity, suppose the circles C_1 and C_2 are each outside the other.

8.2.1 Defining the circles D(p) and E(p)

Let p be a point other than o_1 and o_2.

- Build circle D(p) and its centre d(p) as follows:

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p1 = inv(p, C1) ;
p2 = inv(p, C2) ;
D(p) = circle ( p, p1, p2 ) ;[2]
d(p) = center ( D(p) ) ;[3]
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- Build circle E(p) and its centre e(p) as follows.

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L1 = line ( o1, o2 ) ;
L2 = line ( d(p), p ) ;
L3 = perp ( p, L2 ) ;
e(p) = point ( L1, L3 ) ;
E(p) = circle ( e(p), p ) .
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[3]Suggested tool: Circle through Three Points.
[3]Suggested tool: Midpoint or Center.

8.2.2 Confirm orthogonality

Explain to your own satisfaction why the circles D(p) and E(p) are orthogonal to each other.

8.2.3 Build a Tool for E(p) and D(p)

Keep p, C1, C2 , d(p) and D(p), e(p), E(p) visible, but hide any other construction objects in your figure. Devise a Tool with input objects o1, r1, o2, r2, and p and output objects d(p), D(p), e(p) and E(p).

8.2.4 Build some examples

Use your Tool to construct several pairs of circles, each time using C1 and C2, but for various choices of p. Do not get carried away here, you don't need a lot to learn what you need to learn. Maybe 4 or 5 invocations of the Tool (8 or 10 circles) should be enough. Spread them around by arranging the positions of the various "p" points.

Warm up questions to answer for yourself:

See that there are two pencils of circles. One pencil given by the circles E(p) for any given p, and the other by circles D(p) for any given p.

Aside from the problem that the above construction might break if p lies on C1 or C2, convince yourself that C1 and C2 are members of the pencil of circles of the form E(p) of corresponding to positions of p that lie on C1 or C2.

Watch what happens to the two families of circles as you move C1 and C2 to various positions so that C1 and C2 meet in 0, 1 or 2 points.

1. For the case that circles C1 and C2 are disjoint:
   (a) What sort of family is the collection of all the circles D(p)?
   (b) What sort of family is the collection of all the circles E(p)?

(c) Describe the locus of d(p), as p ranges over all possible points in the plane? (Are there any obvious omissions in the possible positions of d(p)?)

(d) Describe the locus of e(p), as p ranges over all possible points in the plane? (Are there any obvious omissions in the possible positions of e(p)?)

(e) Are there any points that are common to all the circles D(p)?

(f) If so, is there a connection between them and the locus of e(p)?

(g) Are there any points that are common to all the circles E(p)?

(h) If so, is there a connection between them and the locus of d(p)?

2. The same as above, but the for the case in which C1 and C2 are tangent to each other.

3. The same as above, but for the case in which C1 and C2 meet in two distinct points.

8.2.5 Orthogonal families *

Start with a clean figure with non-intersecting circles C1 and C2, and two points r and s. Use two applications of the macro you devised in 8.2.3 to create the four circles, D(r), E(r), D(s) and E(s).

- Print your figure and your Construction Protocol. Confirm for yourself that D(r) and D(s) seem to meet at two points, u and v, on the line of centres of C1 and C2.
- Assuming the previous item, or otherwise, Explain why (prove) that u and v are inverses of each other with respect to C1. *
- Using this fact, or otherwise, explain why (prove) D(r) and E(s) are orthogonal to each other for any choice of r and s. *
- There may be some exceptional cases. If so, identify them.

8.3 Inverses of Families of Circles *

Let Σ be a circle whose centre is O and radius is r. Let A be any point interior to Σ. Suppose that A $\neq$ O. Let B = inv(A, Σ).

Let $\mathcal{F}$ be the family of circles that are incident with A and B. The circles in this family all have their centres on the line `pbis(A,B)`, the perpendicular bisector between A and B. Answer these 5 questions. *

1. Construct a circle Γ with centre at B that is orthogonal to Σ. Let s be defined to be the radius of Γ. Give your construction protocol.
2. Show that O and A are inverses of each other with respect to Γ.
   Hint: Use an equation implied by the fact that Γ is orthogonal to Σ.
3. Using the fact that all the circles in $\mathcal{F}$ go through the centre of Γ. what can you say about $\mathcal{F}^{\Gamma}$, the set of all circles that are inverses of circles in $\mathcal{F}$?
4. Consider inversion with respect to Γ. Let p be any point. Prove that:
   - If p is on Σ, then p^{Γ} is on Σ.
   - If p is inside Σ, then p^{Γ} is inside Σ.
   - If p is outside Σ, then p^{Γ} is outside Σ.

8.4 Reflections in the hyperbolic plane

In what follows, we are working in the Poincaré model of the hyperbolic plane.

Let Ω be any circle. The points and lines of the Poincaré model plane will be called P-points and P-lines to distinguish them from ordinary points and lines in the Euclidean plane. The **P-points** and **P-lines** are defined as follows.

- Any point inside (but not *on*) Ω is said to be a **P-point**.
- The interior arc of any circle orthogonal to Ω whose end points are on Ω is said to be a **P-line**. This includes the arcs of circles of infinite radius that are orthogonal to Ω, which happen to be diameters of Ω.

An inversion with respect to a P-line, when restricted to the P-points and P-lines of the model, is called a **P-reflection** or a reflection of the Poincaré model of the hyperbolic plane. If the P-line is an arc of a circle, the map is an inversion with respect to that circle restricted to the P-points. If the P-line is a segment, the map is a reflection with respect to that diameter of Ω.

8.4.1 Lemma *

Using the result in 8.3 (or otherwise), prove this lemma about the real hyperbolic plane:
Lemma 1. *Let o be the centre of Ω. Given any other P-point p, there is a P-line L, so that the P-reflection defined by L maps p to o and o to p. **

8.5 Elliptic pencil and lines on a point *

Using the construction in question 8.3 (or otherwise), explain how to invert any elliptic pencil of circles to a pencil of lines on a point. *

8.6 Hyperbolic pencil and concentric circles *

Using the ideas used in the construction in 8.3 (or otherwise), explain how to invert any hyperbolic pencil of circles to a pencil of concentric circles. *

(*) Items marked with an asterisk should be submitted for marking.