7 Constructing Inverses Part b

7.3 A self inverse construction

7.3.1 The best construction (?).

On a new figure, again let Σ be a circle with centre O, and let P be any point distinct from O. Construct Q as follows.

```
L1 = line ( O, P );
L2 = perp ( O, L1 );
A = meet ( L2, Σ );
B = meet ( L2, Σ \ A );
L3 = line ( A, P );
L4 = perp ( B, L3 );
Q = meet ( L4, L1 ).
```

7.3.2 Exploration

As with previous constructions, experiment with the location of P, inside and outside Σ, to experience directly and become familiar with various qualitative properties of inversion. For example, when P is inside, Q is outside, and the other way around. Also convince yourself that as P gets closer to O, Q gets much, much farther away, and the other way around. We know that if P is on Σ so is its inverse. Confirm, to your own satisfaction, that this construction exhibits this behaviour.

7.3.3 Comment

This construction might be the shortest one, both in terms of the number of steps, and in terms of its computational complexity.

7.3.4 The reversibility of this construction

Consider now, any position of P, and the corresponding position for Q and the points and lines we used to construct it. Imagine that you start your construction with Q instead of with P, and see that you can perform the steps listed above without adding any new points or new lines.

Recall that the inverse of P with respect to a circle is defined in terms of the ray(O,P) and the radius of the circle. If we had defined the inverse of P to be the point Q as given by this construction, the above argument shows that P equals the inverse of the inverse of P.

7.3.5 (*) Detail

(*) Considering the construction given in 7.3.1, and the lines L3 and L4 as defined there, use what you know about angles in a circle to prove that the point L3∩L4 is on Σ.

7.3.6 Create a Tool

Use the figure you created in 7.3.1 to define a "Tool" or macro called **Inverse point** that uses a point and a circle as input objects and gives the inverse of a point with respect to the circle as the result.

Experiment with a new point R, and use your macro to construct R′, the inverse of R with respect to Σ . Verify that when R is close to P, R′ is close to Q, and when R is close to Q, R′ is close to P.

7.4 The inverse of a line

7.4.1 (*) Construction

On a new, blank figure construct a circle Σ with centre O. Let A and B be two new points, not on O, and construct the line L1 =line (A, B). Move A or B or both if necessary so that L1 is not on O. Use the "Parallel" tool to construct the line L2 through O parallel to L1. Let X be a point on L1. Use the built-in tool called "Reflect Point about Circle" to find Y, the inverse of X with respect to the circle C. Use the tool **Locus** to find the locus of Y as a function of X[1].

(*)Submit your figure together with its Construction Protocol.

7.4.2 Observations

Notice that as you move X on the line AB that Y moves along the locus.

Notice that as you move A and B around, the entire locus moves and so does the line L2.

Notice that the locus appears to be a circle tangent to L2 at O.

Explore the cases for which L1 meets the circle C in 0, 1 or 2 points. In particular, pay attention to those extremal cases for which L1 is far from the circle, and those cases for which L1 is very near O, the centre of C.

7.5 The inverse of a circle

7.5.1 Construction of Locus (*)

In this item we work with the inverse of a circle C_2, with respect to a given circle Σ. Our first effort is to consider all the points X on C_2 and to find their inverses with respect to C.

On a new figure, construct a circle C with centre O. Let C_2 be another circle anywhere in the plane, as long as O does not lie on C_2. Let X be any point on C_2. Use the tool "Reflect Point about Circle" to construct the point Y, the inverse of X with respect to C. Construct the locus of Y as a function of X. Submit your figure, together with its Construction Protocol.

7.5.2 Observations (*)

Explore what happens as C_2 is adjusted to various locations and sizes. Consider locations of C_2 for which C_2 is

[1]Given a track (such as the circle C, the **locus** of Y as a function of X is the set of all points f(X) for which X is on the track.

1. remote from, and outside of, C.
2. close to C
3. tangent to C,
4. meets C in two points,
5. is inside C,
6. is very close to O,
7. is on O,
8. is concentric with C (that is, C and C_2 have the same center).

Summarize your observations in items 3, 4 and 7 in 60 words or less. (*)

7.5.3 An algorithm for the inverse of a circle

The locus tool is not a completely satisfactory solution because it uses the inverse algorithm for a large number, maybe hundreds of points. Your mission is to find an algorithm that uses the inverses of a small number points to find the inverse of C_2 with respect to C.

Given a circle C with center O, and a second circle C_2 with centre O_2, devise a construction for the inverse of C_2 with respect to C that does not involve finding the inverse of every point on C_2.

(When you get it right, you will be able to confirm that your algorithm gives the correct result by comparing it with the locus result or by use of the tool **Reflect Object about Circle**.)