6 The Pascal Configuration: An Exploration

Use the GeoGebra tool called “Conic through 5 points” to construct a conic named Γ determined by five distinct points, no 3 collinear. It is suggested that to avoid a figure that goes off the page, you arrange them so that Γ is an ellipse, and to avoid congestion later, it is suggested that you hide these five points that define the ellipse. (With the GeoGebra page open, in the menu bar, select the drop-down menu called **Edit** and then select **Object Properties**. In that control panel that appears, de-select (uncheck) the item **Show Object** for these five objects.)

Let A, B, C, D, E and F be six *more* distinct points on the conic Γ, so that you may move them around on Γ without moving the conic Γ itself.

Notice that the two hexagons $ABCDEF$ and $BCDEFA$ each have exactly the same set of sides AB, BC, ... , EF and FA, but they are listed in a different order. The same is true for the hexagon $FEDCBA$, but **not** true for the hexagon $BACDEF$ because the latter one has the sides BF and AC and the previous three do not.

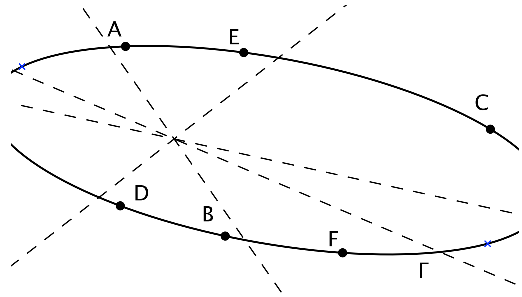
6.1. (*) Identify 12 distinct permutations of the symbols A, B, C, D, E and F so the hexagon produced by each is the same as the hexagon produced when the ordering is A, B, C, D, E, F.

- Explain why each of the 12 permutations gives sides and points that lie on the same Pascal line.
- Explain why there are at most 60 different Pascal lines defined by all the hexagons obtained by resequencing the order in which the six points are used.

6.2. Arrange the six points A through F on Γ so that the three cross points $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ are inside Γ. Highlight the Pascal Line for this conic.

Hint: Consider the Pascal Line of the hexagon $ABFDEC$, (which is obtained from the hexagon given in item 6.1 by swapping the points C and F. This gives us a second Pascal Line on the point $AB \cap CD$. Next, instead of swapping points in the pair $\{C, F\}$, use the pair $\{D, E\}$ to get a third Pascal line on $AB \cap CD$. You will be able to select another pair of points to swap that will give a fourth Pascal Line on the same cross-point $AB \cap CD$.

6.3. (*) Finding Pascal Lines: Show that there are at least four Pascal lines on the cross-point $AB \cap DE$. Submit a figure showing the four lines and your GeoGebra "Construction Protocol."



6.4. (*) Determine the total number of cross points obtained by taking any quadrangle W, X, Y and Z from the set $\{A, B, C, D, E, F\}$ and then finding the cross-point $WX \cap YZ$. Explain your work.

Hint: Count the number of ways to select a quadrangle from $\{A, B, C, D, E, F\}$ and then count the number of cross points or each quadrangle.

6.5. (*) Given any hexagon with six distinct points on a non-degenerate conic, the collection of all cross points and all Pascal lines is said to be a triple system, because each Pascal line is incident with three cross points. Let v be the total number of cross points and b be the number of Pascal lines (blocks). Let k be the number of cross points each Pascal line and r be the number of Pascal lines on each cross point. Verify that

$$vr = bk.$$

This tells us, that in this triple system of Pascal lines and Steiner points, the number of points times the number of lines per point equals the number of lines times the number of points per line.

This triple systems is said to be a **Steiner triple system** in honour of Jacob Steiner (Swiss, 1796 to 1863) who is credited with being the first to have written about them.

6.6. (*) Use GeoGebra to demonstrate that there is a point S_e, that is common to these three lines: [1]

the Pascal Line on the hexagon	A	B	C	D	E	F,
the Pascal Line on the hexagon	A	D	C	F	E	B,
the Pascal Line on the hexagon	A	F	C	B	E	D.

This result was known to Steiner and we think it dates from about 1849.

(*)Submit a figure and the construction protocol.

[1]Notice that the positions of the symbols A, C and E to in the first third and fifth positions do not change but the points B, D and F in the second, fourth and sixth positions in the way that follows the three permutations (BDF), (FBD) and (DFB). The three corresponding hexagons have three distinct coincident Pascal Lines. The subscript "e" in S_e stands for the fact that the given permutations are "even".

Hint: Let S_e be the point of intersection of two of these lines and use the Membership feature of Cabri Geometry II to confirm that S_e is on the third line. This sort of demonstration is not, in my opinion, a proof. *However*, it can provide enough experimental evidence to make it worth our while to try to prove the result. You are not asked to provide the proof.

6.7. (*) Move the point A around the conic Γ and watch the point S_e (as defined in 6.6 above) move about. Plot the locus of the point S_e as a function of the point A. Write a conjecture about the locus of S_e as a function of A. Move the points B, C, ... F to see if your conjecture holds up. If you conjecture does not hold up to the visible evidence, revise your conjecture.[2]

6.8. Similarly, for your own edification and to satisfy your curiosity, discover whether or not there is another point S_o that is on the three lines:[3]

the Pascal Line on the hexagon $A \quad B \quad C \quad F \quad E \quad D$,
the Pascal Line on the hexagon $A \quad D \quad C \quad B \quad E \quad F$,
the Pascal Line on the hexagon $A \quad F \quad C \quad D \quad E \quad B$.

There is a result about S_o, analogous to the one about S_e.

Almost certainly Jacob Steiner knew about these two points S_e and S_o, but we think he might not have known about their loci as a function of A (or any one the five other points B, C, D, E and F, because we have not seen any reference to such a locus in his famous "*Treatise on Conic Sections*".

Submit items marked with "(*)" for marking.

[2]In order to view the desired result, it may help to:
(a) Show all six lines connecting the two points of $\{C, E\}$ to the three points of $\{B, D, F\}$.
(b) Hide all other lines.
(c) Bring points C and E moderately close together.
(d) Move B, D and F somewhat apart. This should allow you to focus on the relevant objects.

[3]The three odd permutations (DFB), (BDF) and (FBD) are used to produce a second set of three distinct coincident Pascal lines. The subscript "o" in S_o stands for "odd".