5 Conics, poles and polars

5.1 (*) A particular conic

Let Γ be the conic whose homogeneous equation is

$$16{x_1}^2 - 4{x_2}^2 - 10x_1x_3 - 6x_2x_3 = 0.$$

1. Write the 3 by 3 symmetric matrix A that represents Γ.
2. For $S : \mathbf{s} = (1, 2, 0)$ and $Q : \mathbf{q} = (1, 2, 1)$, calculate the values of $\mathbf{s}A\mathbf{s}$, $\mathbf{s}A\mathbf{q}$, $\mathbf{q}A\mathbf{s}$, and $\mathbf{q}A\mathbf{q}$.
3. Explain why $S \in \Gamma$ and why $Q \notin \Gamma$.
4. Prove that Γ is non-degenerate.
5. The line SQ meets Γ in a second point T that is distinct from S. Find the coordinates of T.[1]
6. Find the homogeneous coordinates of S^π and T^π, the the polar lines of S and T with respect to Γ.
7. Find the homogeneous coordinates of the tangent line at T.
8. Find the coordinates of the point $R := S^\pi \cap T^\pi$.
9. Calculate the coordinates of R^π, the polar line of R with respect to Γ.
10. Verify that S, T and Q lie on R^π.
11. Using the substitutions $x = x_1/x_3$ and $y = x_2/x_3$ write a Cartesian equation of this conic.

[1]Hint: Let $X : \mathbf{x}$. If $X \in SQ \cap \Gamma$, then $\mathbf{x} = \lambda_1\mathbf{s} + \lambda_2\mathbf{q}$ and $\mathbf{x}A\mathbf{x} = 0$. These two equations give a quadratic polynomial equation in λ_1 and λ_2. Solve it to discover T.

5.2 Poles and polars and tangents

5.2.1 Theory

Now suppose that Γ is any non-degenerate conic and let P be any point not on Γ. Let A and B be any two distinct points on Γ so that A, B and P are *not* collinear. Let C and D be the second points of intersection, respectively, of lines PA and PB with Γ. Let $Q = AD \cap BC$ and $R = AB \cap CD$. If P is an exterior[2] point, let S and T be the points of $QR \cap \Gamma$. The lines PS and PT are tangent to Γ at S and T, respectively.

5.2.2 Practice

1. (*) Do the above construction with GeoGebra, using a circle named Γ in the Euclidean plane as an example of a conic.

   By moving the point P about, verify for yourself that the lines PT and PS appear to be tangent to Γ, as long as P is outside the circle. By moving the points A and B, verify for yourself that the points S and T seem to be stable and independent of the positions of A and B on Γ. Submit a figure and the GeoGebra **Construction Protocol**.

2. (*) Again for a circle with centre O, construct L1 = line(O, P), and find the line L2 perpendicular to L1 at O. Let A and B be the points where L2 meets the Γ. Complete the construction for these two points A and B as described in the paragraph 5.2.1.

   Notice that our algorithm now calls for the use of a point at infinity. Compensate for this by constructing one more line perpendicular to L1. Explain how this compensates for the fact that in the Euclidean plane the two lines do not meet. Can this construction be simplified? (Explain.)

Items marked with and asterisk (*) are to be submitted.

[2] The phrase "*P is exterior to Γ*" means that P^π (the polar line of P) meets the Γ in two points.