

4 Functions and Harmonic sets

4.1 Functions of the form $(ax + b)/(cx + d)$

The purpose of this exercise is to complete the proof that the cross ratio $(x_1, x_2; x_3, x_4)$ is invariant under the map f defined by

$$f : x \mapsto (ax + b)/(cx + d) \quad (1)$$

where $ad - bc \neq 0$.

It is quite easy to see that three special cases of this form have the property that the cross ratio is invariant under their application. They are the functions which have any one of these three forms:

$$x \mapsto x + k, \quad (2)$$

$$x \mapsto mx, m \neq 0, \text{ and} \quad (3)$$

$$x \mapsto 1/x. \quad (4)$$

If we can show that f is a composition of functions of this form, then we will know that the cross ratio is invariant under f . That is, that

$$(f(x_1), f(x_2); f(x_3), f(x_4)) = (x_1, x_2; x_3, x_4).$$

Let four numbers $a, b, c, d \in \mathbb{R}$ be given so that $ad - bc \neq 0$ and let $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Suppose we are given a map $f : \mathbb{R}^* \rightarrow \mathbb{R}^*$ defined by $x \mapsto (ax + b)/(cx + d)$.

$$\begin{aligned} f : \mathbb{R}^* &\rightarrow \mathbb{R}^* \\ x &\mapsto (ax + b)/(cx + d) \end{aligned} \quad (5)$$

4.1.1 Composition

Show that f , as given by equation (5), may be expressed as a composition of a sequence of maps, f_1, f_2, \dots, f_n so that each is one of the types listed in equations 2, 3 or 4. and $f(x) = f_n(\dots f_2(f_1(x)) \dots)$.

4.1.2 Infinity

For the function f what is the image of ∞ ? (Find y so that $y = f(\infty)$.)
What is the pre-image of ∞ ? (Find x so that $f(x) = \infty$.)

4.1.3 Invertibility

For the numbers a , b , c and d , are there any conditions or special cases for which any of the functions in your *your* sequence of functions is not invertible? What are they? Can any of the conditions be overcome by dealing with special cases? If so, how can they be repaired?

4.2 Harmonic Sets

4.2.1 Experience

Not to hand in. Get experience points.

Let A , B , and C be any three points on a line L . Develop a GeoGebra figure that constructs the harmonic conjugate of C with respect to A and B . In this figure you will have to use some point that are not on L . Convince yourself (by moving them around) that your construction is independent of the position of any points you used that are not on the line L . Explore various arrangements of A , B , and C . Consider (i) C to the left of both A and B , (ii) C between A and B (iii) C to the right of both A and B . Do similar tests for B relative to A and C , and so on.) To sharpen your intuition learn for yourself about how D lies relative to A , B and C .

Experiment with your figure to find what happens to D if C is

1. very close to A .
2. very close to B .
3. at the midpoint of A and B .
4. near the midpoint of A and B .
5. very far away from A and B .

6. the ideal point on L .
7. Pick a position of C and notice where D is. What happens to D if you move C to where D was?

Swap the positions of A and B and repeat the above 7 experiments.

4.2.2 Reversibility of harmonic conjugates

Let A , B , and C be any three distinct points. Prove that these two statements are equivalent:

- D is the harmonic conjugate of C with respect to A and B .
- C is the harmonic conjugate of D with respect to A and B .

4.2.3 Two parts of the line

The points A and B separate the projective line into two parts. Is it ever the case that both C and D are in the same part? Give a proof for your answer. Hints: (i) Consider the sign of $(a - x)/(x - b)$. (ii) Consider the definition of harmonic conjugates.

4.3 Harmonic sets are everywhere

4.3.1 The playing field

Imagine a rectangular playing field on a large open plane. Imagine a photograph of the field taken from a high point with a direction and an angle so that in the photograph both the field and the horizon are visible and so that, in the photo, the lines of the parallel sides of the field when extended converge on the line that is the image of the horizon.

Imagine, in the photo, two more lines that would correspond to the diagonals of the field. Each line joins one corner to its opposite corner. Imagine these lines also extended to the horizon, just like the other four lines that are the parallel sides of the field.

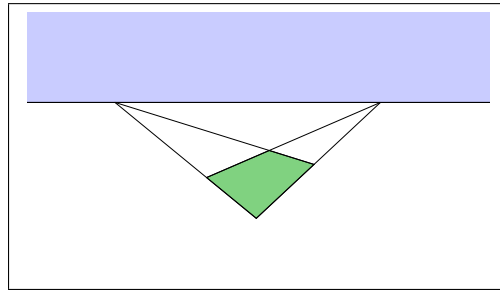


Figure 1: A rectangular playing field

Show that these six lines from the field meet the horizon in four points that form a harmonic set of points. **Submit** a sketch, hand drawn, if you wish. (Artists understood vanishing points and perspective drawings before mathematicians formalized the study of projective geometry).

4.3.2 The general quadrangle

Recall that a frame of reference in the plane consists of four points, no three of which are collinear. When taken two at a time, these four points determine six distinct lines. Show that, given any frame of reference in the plane, there are three lines $h_i, i = 1, 2, 3$ such that each of them meets the six lines determined by the frame in a harmonic set of points.