Lecture Note 01
Collinear points

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Definition 1 (**Collinear).** Suppose three points A, B and C are given. We say that they are **collinear** if there is a line L so that all three points are on L.

Using this definition, we prove the following theorem.

Theorem 2. *Three distinct points are collinear if and only if the coordinates of any one of them is a linear combination of the other two.*

Proof. ($\Longrightarrow$) Suppose A, B and C are three distinct points and suppose they are collinear on the line L. Let the coordinates of these four objects be given by:

$$\begin{array}{ll} \text{A:} & \mathbf{a} = (a_1, a_2, a_3), \\ \text{B:} & \mathbf{b} = (b_1, b_2, b_3), \\ \text{C:} & \mathbf{c} = (c_1, c_2, c_3) \text{ and} \\ \text{L:} & \mathbf{t} = [\ell_1, \ell_2, \ell_3]. \end{array}$$

Since all three points A, B and C are on the line L, we have

$$\mathbf{t} \circ \mathbf{a} = [\ell_1, \ell_2, \ell_3] \cdot (a_1, a_2, a_3) = 0 \tag{1}$$

$$\mathbf{t} \circ \mathbf{b} = [\ell_1, \ell_2, \ell_3] \cdot (b_1, b_2, b_3) = 0 \tag{2}$$

$$\mathbf{t} \circ \mathbf{c} = [\ell_1, \ell_2, \ell_3] \cdot (c_1, c_2, c_3) = 0 \tag{3}$$

Let M be the matrix whose columns have the entries of vectors **a**,**b** and **c**. The above three equations can be written as a vector-matrix equation,

$$\mathbf{t}\, M = [\ell_1, \ell_2, \ell_3] \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0. \tag{4}$$

Equation (4) tells us that **t** satisfies the vector-matrix equation $\mathbf{x}M = 0$. Since there is a non-zero solution (it is **t**) of this equation, M is singular and hence there is also a non-zero solution to the matrix-vector equation $M\mathbf{y} = 0$.[1] Thus, there is a vector $\mathbf{y} = (u, v, w)$ so that $\mathbf{y} \neq \mathbf{0}$ and

$$M\mathbf{y} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{a}u + \mathbf{b}v + \mathbf{c}w = 0. \tag{5}$$

We will show that all three entries in (u, v, w) are non-zero. To do this, we suppose the opposite, that some coordinate is zero. For argument sake, suppose that $u = 0$. We will show that this leads to a contradiction. Even though $u = 0$, we still have $(u, v, w) = (0, v, w) \neq 0$, and hence, at least one of v or w must be non-zero. Suppose it is v. Then we have $0 \neq \mathbf{b} = (-w/v)\mathbf{c}$. Because **b** and **c** are both non-zero, we see that $w \neq 0$ and hence points B and C are equal, which contradicts the assumption that B and C are distinct. Hence $u \neq 0$.

Similar arguments show that $v \neq 0$ and $w \neq 0$.

Because all three entries a, b, and c are non-zero, not only are **a**, **b** and **c** linearly dependent, but we may solve for any one of them as a linear combination of the other two.

This completes the proof that three distinct collinear points have coordinate vectors that are linearly dependent. □

Proof. ($\Longleftarrow$) The proof in the opposite direction, that linear dependence of coordinate vectors implies that three points are collinear, is left as an exercise for the reader. □

Exercise 3. Prove the converse of what has been shown so far, namely, that if three distinct points are linearly dependent, there is a line that is incident with all three.

Corollary 4. *Three distinct* lines *are coincident if and only if their coordinate vectors are linearly dependent.*

Exercise 5. Let three points in the plane be given with homogeneous coordinates (2, 5, 0), (0, 3, 2) and (-2, 1, a). (a) Find homogeneous coordinates for the line on the first two of three points. (b) Show that for the three points to be collinear, we must have $a = 4$.

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[1] There is a non-trivial solution to the equation $\mathbf{x}M$=0 $\iff$ M is singular $\iff$ the equation $My = 0$ has a non-trivial solution.