Notes about Desargues Theorem

Spring Term, 2013

1 The Theorem of Desargues

Definition 1 (Perspective from a point)**.** Let V be a point and let two triangles be given so that their vertices are distinct from V. We say that the two triangle are in perspective from V if the triangles can be labeled so that vertices A_1, B_1 and C_1 are on one triangle and vertices A_2, B_2 and C_2 are on the other and

- the three points V, A_1 and A_2 are collinear,
- the three points V, B_1 and B_2 are collinear, and
- the three points V, C_1 and C_2 are collinear.

See Figure 1.

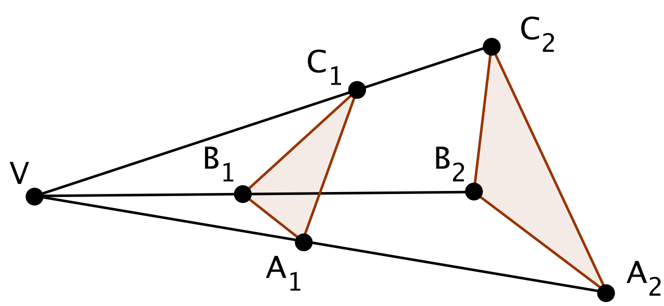

Figure 1: Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in perspective from the point V. (See Definition 1.)

Definition 2 (Perspective from a line)**.** See Figure 2. Two trilaterals with sides a_1, b_1 and c_1 and sides a_2, b_2 and c_2 are said to be in perspective from a line v if

- the three lines v, a_1 and a_2 are coincident,
- the three lines v, b_1 and b_2 are coincident, and

- the three lines v, c_1 and c_2 are coincident.

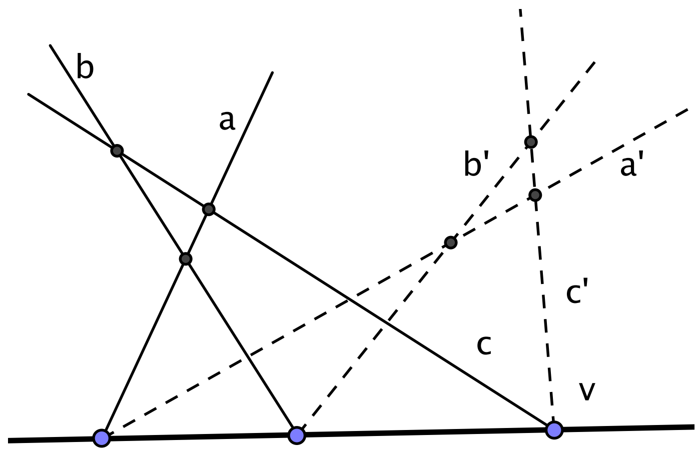

Figure 2: Trilateral a', b', c' (with dotted lines) and trilateral a, b and c are in perspective from the line v (the emboldened line). (See Definition 2.)

Theorem 1 (Desargues'). *Two triangles are perspective from a point if and only if they are perspective from a line.*

Proof (Part 1, $\Longrightarrow$).
We start by showing that having two triangles perspective from a point implies they are perspective from a line. Suppose that triangles $A_1B_1C_1$ and $A_2B_3C_2$ are in perspective from a point V so that V be the point common to the three lines A_1A_2, B_1B_2 and C_1C_2.

For the rest of this note, let the two triangles A_i, B_i and C_i for i= 1,2, have sides $a_i = line(B_i, C_i)$, $b_i = line(C_i, A_i)$, and $c_i = line(A_i, B_i)$, for i=1,2, respectively.

For a frame of reference select the points A_1, B_1, C_1 and V, in that order. Using the fundamental theorem, we may select their coordinates to be $A_1 : (1,0,0), B_1 : (0,1,0); C_1 : (0,0,1)$; and $V : (1,1,1)$.

Since A_2 is on the line VA_1 we have $A_2 : \lambda_1(1,0,0) + \lambda_2(1,1,1)$. Since A_2 is distinct from A_1 and from V, neither λ_1 nor λ_2 may be zero. If we let $a = \lambda_1/\lambda_2$, we write $A_2 : (a+1,1,1)$ where $a \neq 0$. Similarly, $B_2 : (1,1+b,1)$ for some $b \neq 0$ and $C_2 : (1,1,1+c)$ where $c \neq 0$.

Our next step is to find the intersection $A_1B_1 \cap A_2B_2$.

The line A_1B_1 is represented by line coordinates [0, 0, 1] because the vector [0, 0, 1] is the only non-zero vector (up to a non-zero scalar multiple) whose dot product with both A_1 and B_1 are both zero.

The coordinates of the line A_2B_2 are found by the vector cross product

$$[1+a,1,1]\times[1,1+b,1]$$

or by any non-zero solution of the matrix vector equation:

$$\begin{pmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \end{pmatrix}\mathbf{x}=0$$

By either of these methods, we have the intersection

$$A_1B_1 \cap A_2B_2 : [a,-b,0].$$

Similar work gives

$$B_1C_1 \cap B_2C_2 : [0,b,-c]$$

$$C_1A_1 \cap C_2A_2 : [-b,0,a].$$

Notice that all three of these points lie on the line $[a,b,c]$. Call this line v. Since the point $A_1B_1 \cap A_2B_2$ lies on the line v, the three lines A_1B_1, A_2B_2 and v are coincident. Similarly, the three lines B_1C_1, B_2C_2 and v are coincident, and the three lines C_1A_1 , C_2A_2 and v are coincident. This says that the two triangles are perspective from the line v. That is, v is the axis of perspectivity for the $A_1B_1C_1$ and $A_2B_2C_2$.

This completes the proof that two triangles that are perspective from a point are also perspective from a line. $\square$

Proof (Part 2, $\Longleftarrow$).
Here the job is to show that if two triangles are in perspecive from a *line*, they are perspective from a *point*. The principal of duality helps us here. The definitions of perspective from a point and perspective from a line are duals of each other, because we may interchange the words point by line and line by point and still have a valid theorem. $\square$

Proof (Part 3, $\Longleftarrow$).
In this part we give a second proof of the converse. It is quite awesome, in my opinion, to use the obverse to prove the converse. The difference comes with which triangles we use to be in perspective from a point.

Let us continue with the nomenclature used in Part 1, but without assuming that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are are in perspective from a point.

Conversely assume that the two triangles are are in perspective from the line v and show that they are in perspective from a point. We do not assume that triangles $A_1B_1C_1$ and $A_1B_2C_2$ are in perspective from a point such as v. This will, however, be shown. See Figure 3.

Consider the three pairs of lines:

$$A_1B_1 \text{ and } A_2B_2 \text{ meeting at } Z, \tag{1}$$
$$B_1C_1 \text{ and } B_2C_2 \text{ meeting at } X, \text{and} \tag{2}$$
$$C_1A_1 \text{ and } C_2A_2 \text{ meeting at } Y. \tag{3}$$

Consider the possibility that Z might be the centre of perspectivity for some pair of triangles. Indeed it is. See Figure 3.

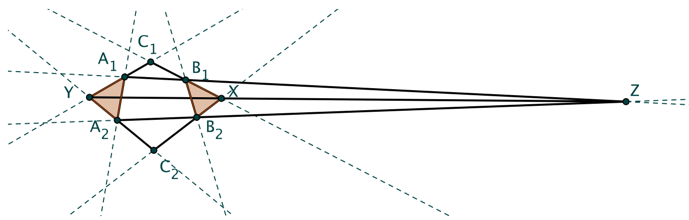

Figure 3: The two triangles $A_1B_1C_1$ and $A_2B_2C_2$ in relation to the point Z.

Because the sides of triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in perspective from from the line XYZ, the triangles A_1A_2Y and B_1B_2X are in perspective from the point Z. We now apply the proof of part 1 of the theorem that two triangles perspective from a point are perspective from a line. The difference is that we apply the proven portion of the theorem to the triangles A_1A_2Y and B_1B_2X.

By the first proof (Part 1, $\Longrightarrow$), we know that these same triangles are perspective from some axis somewhere. That axis of perspectivity must be the line C_1C_2 and hence must contain the point of intersection $A_1A_2 \cap B_1B_2$. That tells us that the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in perspective from the point common to the three lines A_1A_2, B_1B_2 and C_1C_2. $\square$