A Note about
Harmonic Conjugates

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1 Introduction

This note is not so much about showing that a certain set of four points forms a harmonic set, but about a comparison of three different constructions all of which, if we choose, may use exactly the same set of auxiliary points and lines. All three of these constructions may be thought of in a way that is independent from one another, but each may be used as a model for the other two, using exactly the same points and lines, but arrived at through a different (but related) sequence of construction steps.

The latter two sequences of construction steps demonstrate a kind of reciprocity, each have the same outline but they exhibit duality that connects the second and third constructions which may be thought of as having a kind of reciprocal relationship, namely that D is the harmonic conjugate of C with respect to A and B if and only if C is the harmonic conjugate of D with respect to A and B. That is,

$$D = H(A,B;C) \iff C = H(A,B;D).$$

We use the notation $(A,B;C,D) = -1$ to mean that the cross ratio of A and B with respect to C and D is -1. The notation $H(A,B;C,D)$ is used to mean that A, B, C, D (in that order), form a harmonic set, and $D = H(A,B;C)$ to mean D is the harmonic conjugate of C with respect to A and B. Then

$$D = H(A,B;C) \iff H(A,B;C,D) \iff (A,B;C,D) = -1.$$

In this note, three different construction sequences are given.

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Comparing construction sequences one and two we will see that the same set of points and lines may be used in two different ways to produce the same result, in these two cases that $D = H(A,B;C)$.

Comparing constructions two and three, we will see that the same set of points and lines can be used to produce different results, in this case $D = H(A,B;C)$ and $C = H(A,B;D)$.

2 The first construction: $D = H(A,B;C)$

We start with three points, A, C, and B in that order, left to right, on a line L, and we construct a constellation of four points, P, Q, R and S off the line L to produce $D = H(A,B;C)$.

Here is the first construction. With A, C, and B arranged consecutively, left to right on L, the quadrangle P, Q, R, S is constructed with the points being chosen in that order. If Q is selected so that it appears between P and A in the figure, then the remaining points of the quadrangle Q, R and S appear in a counterclockwise manner. This might be the only reason to remember this particular construction.

Construction Sequence 1

```
    P  = point ( \L ) ;
1.  PA = line ( P, A ) ;
    Q  = point ( PA \A, P ) ;
2.  PC = line ( P, C ) ;
3.  BQ = line ( B, Q ) ;
    R  = point ( PC, BQ ) ;
4.  AR = line ( A, R ) ;
5.  PB = line ( P, B ) ;
    S  = point ( PB, AR ) ;
6.  QS = line ( Q, S ) ;
    D  = point ( L, QS ) .
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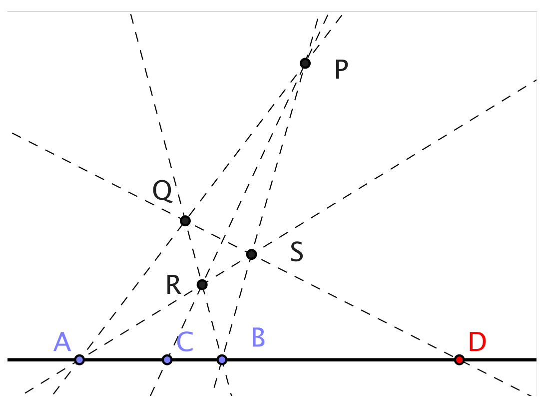

Preparation for the next construction.

Consider the image above, keeping the line L the three points A, C, and B on L and the four points P, Q, R, S that are not on L.

In the second construction we again construct the six lines, but in a different way. Ordinarily, our construction might start with something like this: "Pick any point P not on L and draw the line PA", but because the result of the construction is independent of the position of P, it is good enough to use the same P selected in the previous construction. Similar comments apply to Q, R, and S.

For this reason, in the second construction, we will use exactly the same points and lines, but they will be selected and constructed in a different order and may be defined by a different pair of points that are known to be already on the line.

Yes, we could select a fresh set of points, of course, but for our purposes in comparing the second and third construction sequences, it is useful to work with the same set of points in set 3 as in set 2, so why not use the same points from set 1 for all three?

3 The second construction: $D = H(A, B; C)$ (again)

Construction Sequence 2, using the same points: A, C, B on L and P, Q, R, S not on L.

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    P  = point ( \L ) ;
1.  CP = line ( C, P ) ;
    R  = point ( PC \P, C ) ;
2.  AP = line ( A, P ) ;
3.  BR = line ( B, R ) ;
    Q  = point ( AP, BR ) ;
4.  AR = line ( A, R ) ;
5.  BP = line ( B, P ) ;
    S  = point ( AR, BP ) ;
6.  QS = line ( Q, S ) ;
    D  = point ( L, QS ) .
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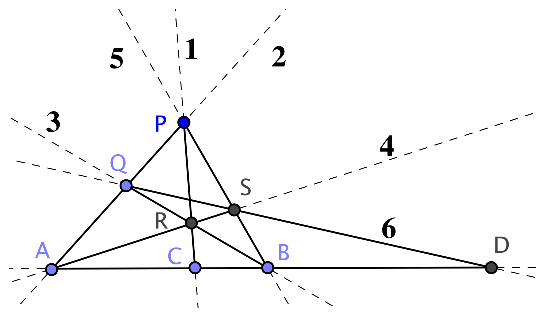

Overview
The second construction for $D = H(A, B; C)$ may be thought of this way:

- Pick P and R on line (not L) through C.
- Use A, B, P and R to construct S and Q.
- Use S, Q and line L to construct D.

In the next construction, for $C = H(A, B; D)$ may be thought of this way:

- Pick Q and S on a line through D.
- Use A, B, Q and S to construct P and R.
- Use P, R and the line L to construct C.

The two previous paragraphs demonstrate a reciprocal relationship between the pair $\{P, R\}$ which are collinear with C and the pair $\{Q, S\}$ which are collinear with D.

4 The third construction: $C = H(A, B; D)$

Construction Sequence 3

```
    Q  = point ( \L ) ;
1.  DQ = line ( D, Q ) ;
    S  = point ( PC \P, C ) ;
2.  AQ = line ( A, Q ) ;
3.  BS = line ( B, S ) ;
    P  = point ( AQ, BS ) ;
4.  AS = line ( A, S ) ;
5.  BQ = line ( B, Q ) ;
    R  = point ( AS, BQ ) ;
6.  PR = line ( P, R ) ;
    C  = point ( L, PR ) .
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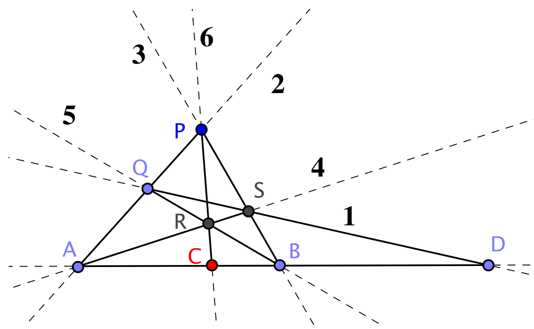

5 Summary

Our objective here was to show a way to think about the selection and naming of the four points so as to reveal a kind of duality in the *construction steps* for $D = H(A, B; C)$, and the reciprocal construction of the harmonic conjugate of $C = H(A, B; D)$, making use of the *same* four auxiliary points P, Q, R and S.

The beauty here is that from the constellation of auxiliary points { P, R} and {Q, S} one has a geometric demonstration that $D = H(A, B; C) \iff C = H(A, B; D)$.